

# CONTEST CORNER SOLUTIONS

*Statements of the problems in this section originally appear in 2016: 42(7), p. 291–293.*

**CC231.** If  $x^2 + y^2 = 6xy$  with  $y > x > 0$ , find  $\frac{x+y}{x-y}$ .

*Originally Question 6 of The Seventh W.J. Blundon Contest, 1990.*

*We received 19 submissions of which 15 were correct and complete. We present 2 solutions.*

*Solution 1, by Dan Daniel.*

We have

$$x^2 + y^2 = 6xy \iff 2x^2 + 2y^2 - 4xy = x^2 + y^2 + 2xy \iff 2(x-y)^2 = (x+y)^2,$$

so

$$\left(\frac{x+y}{x-y}\right)^2 = 2.$$

Since  $y > x$ , then  $\frac{x+y}{x-y} = -\sqrt{2}$ .

*Solution 2, by Titu Zvonaru.*

Let  $t = \frac{y}{x}$ . From  $x^2 + y^2 = 6xy$  we obtain

$$t^2 - 6t + 1 = 0.$$

Since  $t > 1$ , we get

$$\frac{x+y}{x-y} = \frac{1+t}{1-t} = \frac{1+3+\sqrt{8}}{1-3-\sqrt{8}} = -\frac{2+\sqrt{2}}{1+\sqrt{2}} = -\sqrt{2}.$$

*Editor's Comments.* All the wrong submissions reported as a result  $\sqrt{2}$  instead of  $-\sqrt{2}$ . Someone did a mistake when copying the problem (wrote  $x > y$  instead of  $y > x$ ), someone forgot that  $y > x$ , so when you take the square root of  $(x-y)^2$  you get a negative number. Konstantine Zelator also considered the general case

$$x^2 + y^2 = kxy, \quad k \text{ is a real number bigger than } 2,$$

and proved that

$$\frac{x+y}{x-y} = -\sqrt{\frac{k+2}{k-2}}.$$

**CC232.** Seven tests are given and on each test no ties are possible. Each person who is the top scorer on at least one of the tests or who is in the top six on at least four of these tests is given an award, but each person can receive at most one award. Find the maximum number of people who could be given awards if 100 students take these tests.

*Originally Team Question 3 of the 1988 Florida Mathematics Olympiad.*

*We received four correct solutions. We present a combination of all four solutions.*

The maximum number of people who could be given awards is 15. There are always 7 top scorers who get an award. The other awards are given to people who were in the top six in at least 4 tests. Altogether 35 people are ranked 2nd, 3rd, 4th, 5th, and 6th. The maximum number of people who could be given awards will be reached if there are as many people who are four times in the top 6 as possible.

$35 = 4 \cdot 8 + 3$ , so the maximum number of people who could get awards by being four times in the top 6 is 8. Thus, the maximum number of students that can be given an award is  $7 + 8 = 15$ ; and this works as long as there are at least 15 test-takers, whereas number 100 does not play any special role. A specific set of outcomes with 15 awards can be realized by

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
1 <sup>st</sup>	A	B	C	D	E	F	G
2 <sup>nd</sup>	H	H	H	H	I	I	I
3 <sup>rd</sup>	I	J	J	J	J	K	K
4 <sup>th</sup>	K	K	L	L	L	L	M
5 <sup>th</sup>	M	M	M	N	N	N	N
6 <sup>th</sup>	O	O	O	O	X	Y	Z

**CC233.** Let  $P$  be a point in the interior of the rectangle  $ABCD$ . Suppose that  $PA = a$ ,  $PB = b$  and  $PC = c$ , find, in terms of  $a, b, c$ , the length of the line segment  $PD$ .

*Originally Individual Question 12 (b) of the 1988 Florida Mathematics Olympiad.*

*We received 13 correct solutions. We present the solution by Titu Zvonaru.*

Let  $P_1$  and  $P_2$  be the projections of  $P$  onto  $AB$  and  $AD$ , respectively. By the Pythagorean Theorem,

$$(PP_1)^2 + (PP_2)^2 = a^2, (AB - PP_1)^2 + (PP_2)^2 = b^2, (AD - PP_2)^2 + (AB - PP_1)^2 = c^2.$$

It follows that

$$(PD)^2 = (AD - PP_2)^2 + (AB - PP_1)^2 = a^2 + c^2 - b^2,$$

so that

$$PD = \sqrt{a^2 + c^2 - b^2}.$$

**CC234.** Find  $B$  if

$$x = \frac{\log_{10} 16/3}{\log_{10} B}$$

is the solution to the exponential equation

$$2^{2x+4} + 3^{3x+2} = 4^{x+3}.$$

*Originally Individual Question 10 of the 1988 Florida Mathematics Olympiad.*

*We received 14 correct solutions and one incorrect solution. We present the solution by Kathleen Lewis.*

The given equation  $2^{2x+4} + 3^{3x+2} = 4^{x+3}$  can be rewritten as

$$3^{3x+2} = 4^{x+3} - 4^{x+2} = 3 \cdot 4^{x+2}.$$

Thus

$$3 \cdot 3^{3x} = 16 \cdot 4^x,$$

so  $27^x/4^x = 16/3$ . Then

$$x \log_{10}(27/4) = \log_{10}(16/3),$$

so  $B = 27/4$ .

**CC235.** Find the area of a regular octagon formed by cutting equal isosceles triangles from the corners of a square with sides of one unit.

*Originally Question 6 of The Ninth W.J. Blundon Contest, 1992.*

*We received 13 correct solutions. We present the solution by Kathleen Lewis.*

To end up with a regular octagon with sides of length  $x$ , each triangle that is cut off will have legs of length  $x/\sqrt{2}$ . The original square was a unit square, so

$$1 = x + 2 \cdot x/\sqrt{2} = x(1 + \sqrt{2}),$$

implying that

$$x = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1.$$

Each triangle that was cut off has area  $x^2/4$ , so the total area removed is  $x^2 = 3 - 2\sqrt{2}$ . Therefore, the remaining area is

$$1 - (3 - 2\sqrt{2}) = 2\sqrt{2} - 2.$$

