# The *pqr* Method: Part I

Steven Chow, Howard Halim and Victor Rong

## Introduction

Each year, students who perform well on the International Mathematics Tournament of Towns are invited to the Tournament of Towns Summer Conference, to spend 10 days in Russia investigating current streams of mathematical research [1]. The three authors were participants of the 2016 Summer Conference, where they learned about the *pqr* method and its applications to solving inequalities.

The *pqr* method is a way to solve symmetric 3-variable inequalities by using three lemmas collectively known as the *pqr* lemmas. Given an inequality in terms of variables $a$, $b$, and $c$, we can make the substitutions $p = a+b+c$, $q = ab+bc+ca$, and $r = abc$. When two of $p$, $q$, $r$ are fixed, the third obtains its maximum and minimum values when two of $a$, $b$, $c$ are equal or one of $a$, $b$, $c$ is 0. This reduces the problem to a 2-variable inequality.

This method is also known as the *uvw* method (although with slightly different variables) which was popularized by Knudsen [2].

## Proving the *pqr* lemmas

The proof of the *pqr* lemma relies on a few lemmas whose proofs are left as an exercise for the reader. For three complex numbers $a$, $b$, $c$, let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$, and define

$$T(p,\,q,\,r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 = (a-b)^2(b-c)^2(c-a)^2.$$

**Lemma 1.** If $p$, $q$, $r$ are real numbers, then $a$, $b$, $c$ are real numbers if and only if $T(p,\,q,\,r) \geq 0$.

**Lemma 2.** $a$, $b$, $c$ are non-negative real numbers if and only if $p,\,q,\,r \geq 0$ and $T(p,\,q,\,r) \geq 0$.

**The *pqr* Lemma.** When we fix two of $p$, $q$, $r$ such that there exists triples $(p,\,q,\,r)$ satisfying $p,\,q,\,r \geq 0$ and $T(p,\,q,\,r) \geq 0$, the unfixed variable obtains its maximum and minimum values when two of $a$, $b$, $c$ are equal. There is one exception – when $r$ is the unfixed variable, its minimum value occurs when either two of $a$, $b$, $c$ are equal, or one of them is equal to 0.

We will prove the $r$-lemma, the case when $r$ is the unfixed variable. The $p$-lemma and $q$-lemma have similar proofs.

*Proof.* Fix $p = p_0$ and $q = q_0$. Then, $T(p_0, q_0, r)$ is a quadratic in $r$. Since its leading coefficient is negative, it points downwards, and the inequality $T(p_0, q_0, r) \geq 0$ defines a closed interval (the interval between both of its roots). The maximum value of $r$ occurs at the right endpoint of the interval. Since $T(p_0, q_0, r) = 0$ at the endpoints, it follows that $(a-b)^2(b-c)^2(c-a)^2 = 0$, so two of $a$, $b$, $c$ are equal.

Since we have a second inequality, $r \geq 0$, the minimum value of $r$ occurs either at the left endpoint, or at $r = 0$ (if the left endpoint is negative). At the left endpoint, $T(p_0, q_0, r) = 0$, so two of $a$, $b$, $c$ are equal. When $r = 0$, one of $a$, $b$, $c$ equals 0. This completes the proof. □

The ability to use the *pqr* lemmas hinges on the property that any symmetric polynomial in terms of $a$, $b$, $c$ can be written in terms of $p$, $q$, $r$. We will prove this property by demonstrating an algorithm applicable to any symmetric polynomial.

*Proof.* Call a symmetric polynomial *expressible* if it can be written in terms of $p$, $q$, $r$. Let $s_k = a^k + b^k + c^k$. First, we prove by induction that $s_k$ is expressible for all non-negative integers $k$.

For the base case, we see that $s_0$, $s_1$, and $s_2$ are expressible, because $s_0 = 3$, $s_1 = p$, and $s_2 = p^2 - 2q$. Since $s_k = ps_{k-1} - qs_{k-2} + rs_{k-3}$, $s_k$ is expressible for all non-negative integers $k$.

We see that

$$\begin{aligned} s_k s_l - s_{k+l} &= (a^k + b^k + c^k)(a^l + b^l + c^l) - (a^{k+l} + b^{k+l} + c^{k+l}) \\ &= a^k b^l + b^k c^l + c^k a^l + a^k c^l + b^k a^l + c^k b^l \\ &= \sum_{sym} a^k b^l. \end{aligned}$$

Therefore, $\sum_{sym} a^k b^l$ is expressible for all non-negative integers $k$, $l$. Finally, note that

$$\sum_{sym} a^k b^l c^m = r^n \sum_{sym} a^{k-n} b^{l-n} c^{m-n},$$

where $k$, $l$, $m$ are positive integers and $n = \min(k, l, m)$. The sum on the right hand side is expressible, so the sum on the left is also expressible. It follows that all symmetric polynomials are expressible. □

## Examples of the *pqr* Method

**Example 1.** Let $a$, $b$, $c$ be non-negative real numbers. Prove that

$$a^5 + b^5 + c^5 + abc(ab + bc + ca) \geq a^2b^2(a + b) + b^2c^2(b + c) + c^2a^2(c + a).$$

*Solution.* Fix $p$ and $q$. Let

$$f(r) = (7p^2 - 3q)r + p^5 - 5p^3q + 4pq^2,$$

then the inequality is equivalent to $f(r) \geq 0$. Since $f$ is linear in terms of $r$, if the inequality holds for the extreme values of $r$, it holds for all values of $r$. From the $r$-lemma, the maximum and minimum of $r$ occur when two of $a$, $b$, $c$ are equal, or one of them is 0. WLOG, we can assume that either $a = b$ or $a = 0$.

When $a = 0$, the inequality is equivalent to $b^5 + c^5 \geq b^3c^2 + b^2c^3$. This inequality can be proved by summing up the two inequalities

$$\frac{b^5 + b^5 + b^5 + c^5 + c^5}{5} \geq b^3c^2 \quad \text{and} \quad \frac{b^5 + b^5 + c^5 + c^5 + c^5}{5} \geq b^2c^3,$$

both of which are true by the AM-GM inequality. When $a = b$, the inequality is equivalent to $c^5 + a^4c \geq 2a^2c^3$ after expanding. This inequality is true by the AM-GM inequality. Since we have proved the inequality when $a = 0$ and $a = b$, then we are done by the $r$-lemma.

Note that this inequality can also be proved by Schur's Inequality:

$$\sum_{cyc} a(a^2 - b^2)(a^2 - c^2) \geq 0.$$

However, the solution using the *pqr* lemma does not require any creative observations and is much easier to come up with.

**Example 2.** Let $a$, $b$, $c$ be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{9 - ab} + \frac{1}{9 - bc} + \frac{1}{9 - ca} \leq \frac{3}{8}.$$

*Solution.* Note that $a$, $b$, $c \leq 3$, so $ab$, $bc$, $ca \leq 9$, and all denominators are positive. Multiplying by $8(9 - ab)(9 - bc)(9 - ca)$ on both sides of the inequality, we obtain

$$\sum_{cyc} 8(9 - ab)(9 - bc) \leq 3(9 - ab)(9 - bc)(9 - ca).$$

Expanding yields

$$243 - 99\sum_{cyc} ab + 19\sum_{cyc} a^2bc - 3a^2b^2c^2 \geq 0$$

which is equivalent to

$$-3r^2 + 19pr - 99q + 243 \geq 0.$$

Note that $p$ is already fixed by the constraint $a + b + c = 3$. Fix $r$ as well. Let $f(q) = -3r^2 + 19pr - 99q + 243$. Since this is linear in terms of $q$, we only need to check the minimum and maximum values of $q$. By the $q$-lemma, these occur when $a = b$.

So it is enough to prove the inequality when $b = a$, $c = 3 - 2a$, and $0 \le a \le \frac{3}{2}$. Substituting these values and expanding yields

$$-4a^6 + 12a^5 - 9a^4 - 38a^3 + 156a^2 - 198a + 81 \ge 0.$$

This is equivalent to

$$-(a-1)^2(2a^2 - 3a + 9)(2a^2 + a - 9) \ge 0$$

which is true since

$$2a^2 - 3a + 9 = a^2 + (a - \frac{3}{2})^2 + \frac{27}{4}$$

and

$$2a^2 + a - 9 \le 2(\frac{3}{2})^2 + \frac{3}{2} - 9 = -3 < 0$$

for $0 \le a \le \frac{3}{2}$.

**Example 3.** Let $a$, $b$, $c$ be non-negative real numbers such that $a + b + c = 4$ and $a^2 + b^2 + c^2 = 6$. Prove that

$$a^6 + b^6 + c^6 \le a^5 + b^5 + c^5 + 32.$$

*Solution.* The conditions are equivalent to $p = 4$ and $p^2 - 2q = 6$. With a little work, the inequality is equivalent to

$$p^6 - 6p^4q + 6p^3r + 9p^2q^2 - 12pqr - 2q^3 + 3r^2 \le p^5 - 5p^3q + 5pq^2 + 5p^2r - 5qr + 32.$$

Note that from the conditions, $p$ and $q$ are already fixed with $p = 4$ and $q = 5$. Let

$$f(r) = -3r^2 + (12pq - 6p^3 + 5p^2 - 5q)r + p^5 - 5p^3q + 5pq^2 + 32 - p^6 + 6p^4q - 9p^2q^2 + 2q^3.$$

Since $f(r)$ is concave in terms of $r$, it is enough to prove the inequality for the minimum and maximum possible values of $r$. By the $r$-lemma, it is sufficient to consider the cases $a = 0$ and $a = b$.

If $a = 0$, there are no real solutions for $b$ and $c$ satisfying the conditions $a+b+c = 4$ and $a^2 + b^2 + c^2 = 6$. If $a = b$, solving the system of equations yields $a = b = 1$ and $c = 2$ or $a = b = \frac{5}{3}$ and $c = \frac{2}{3}$. These values satisfy the inequality, so the inequality is true.

It is important to note that one must be careful of the conditions when applying the $pqr$ lemma. In the previous example, it should be noted that there are only two possible values of $c$ when $a = b$ which obey the given conditions. The reason for this is because $p$ and $q$ were already fixed by the conditions, while usually $p$ and $q$ are arbitrarily fixed. However, this problem is not always so easily resolved. For example, consider the following problem:

> Let $a$, $b$, $c$ be non-negative real numbers such that
>
> $$abc[(a-b)(b-c)(c-a)]^2 = 1.$$
>
> Find the minimum of $a + b + c$.

Even though the function $a+b+c$ is monotonic in terms of $p$, one cannot set $a=b$ or $a=0$ as that leads to a contradiction in the condition. If a problem has a similar issue when setting $a=b$ or $a=0$, then the $pqr$ method cannot be used.

## Problems

Here are a few problems which can be solved using the $pqr$ method in a standard fashion.

**Problem 1.** Let $a$, $b$, $c$ be non-negative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq\frac{2}{1+abc}.$$

**Problem 2.** Let $a$, $b$, $c$ be non-negative real numbers such that $ab+bc+ca\neq 0$. Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\geq\frac{10}{(a+b+c)^2}.$$

**Problem 3 ([3]).** Let $a$, $b$, $c$ be positive real numbers. Prove that

$$\sum_{cyc}\frac{ab(a+b)^2}{(c+a)(c+b)}\geq\frac{(a+b+c)^2}{3}.$$

**Problem 4.** Let $a$, $b$, $c$ be real numbers such that $a$, $b$, $c\geq 1$ and $a+b+c=9$. Prove that

$$\sqrt{ab+bc+ca}\leq\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

*To be continued.*

## References

[1] Doledenok, A., Fadin, M., Menshchikov, A., Semchankau, A. (2016, August). The pqr-method. Retrieved from `http://www.turgor.ru/lktg/2016/3/index.htm`

[2] Knudsen, M.B.T. (2009, May 26) The UVW-method [Online forum comment]. Message posted to `http://www.artofproblemsolving.com/community/c13188h278791`

[3] lebathanh (2016, October 4) inequalities [Online forum comment]. Message posted to `http://artofproblemsolving.com/community/c6h1314627`