SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(5), p. 220–224.

4141. *Proposed by Leonard Giugiuc, Daniel Sitaru and Oai Thanh Dao; modified by the editor.*

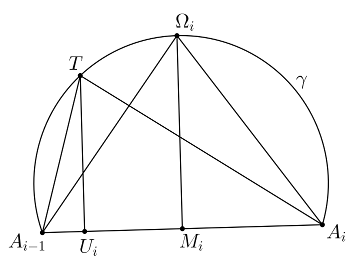
a) Let $A_0A_1\dots A_{n-1}$ be a convex n-gon for which there exists an interior point T such that $\angle A_{i-1}TA_i = \frac{2\pi}{n}, i = 1, 2, \dots n$ (with $A_n \equiv A_0$). Construct regular n-gons Π_i externally on the sides $A_{i-1}A_i$. Prove that

$$[A_0A_1\dots A_{n-1}] \le \frac{1}{n}\sum_{i=1}^{n}[\Pi_i]$$

(where square brackets denote area).

b) $\star$ Does the inequality continue to hold if the given convex polygon is arbitrary?

Three solutions were received for part (a), and all were correct. Part (b) attracted no response, so it remains open. We present the solution to part (a) by Michel Bataille.



Let Ω_i be the reflection in $A_{i-1}A_i$ of the centre O_i of the polygon Π_i (see figure above). Because $\angle A_{i-1}TA_i = \angle A_{i-1}\Omega_iA_i = \frac{2\pi}{n}$, both T and Ω_i belong to the circular arc γ with endpoints A_{i-1} and A_i. Furthermore,

$$\frac{[\Pi_i]}{n} = [A_{i-1}O_iA_i] = [A_{i-1}\Omega_iA_i].$$

Let U_i and M_i denote the projections of T and Ω_i onto the line $A_{i-1}A_i$, respectively. Since Ω_i and M_i are on a diameter of the circle containing γ, we have $TU_i \le \Omega_iM_i$. We deduce that

$$[A_{i-1}TA_i] = \frac{1}{2}A_{i-1}A_i \times TU_i \le \frac{1}{2}A_{i-1}A_i \times \Omega_iM_i = [A_{i-1}\Omega_iA_i]$$

and so

$$[A_0A_1 \dots A_{n-1}] = \sum_{i=1}^{n}[A_{i-1}TA_i] \le \sum_{i=1}^{n}[A_{i-1}\Omega_i A_i] = \sum_{i=1}^{n}\frac{1}{n}[\Pi_i]$$

and the required inequality follows.

4142. *Proposed by Daniel Sitaru.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$\left(1+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \le \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

We received 4 correct solutions. We present the solution by Arkady Alt.

Assuming, due to the homogeneity of the original inequality, that $a+b+c=1$ and denoting

$$p = ab+bc+ca,\ \ q = abc,$$

we obtain

$$a^2+b^2+c^2 = 1-2p,$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = \frac{(a+b)\,(b+c)\,(c+a)}{abc} = \frac{p-q}{q},$$

and

$$1+\frac{a^2+b^2+c^2}{ab+bc+ca} = 1+\frac{1-2p}{p} = \frac{1-p}{p}.$$

The original inequality thus becomes

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{p}{q}-1.$$

Since $0 < q \le \dfrac{p^2}{3}$, we have $\dfrac{p}{q} \ge \dfrac{3}{p}$, and it suffices to prove the inequality

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{3}{p}-1.$$

For $0 < p \le \frac{1}{3}$, this is successively equivalent to

$$\frac{1-p}{p} \le \left(\frac{3-p}{p}\right)^{1-2p},$$

$$\left(\frac{3-p}{p}\right)^{2p} \le \frac{3-p}{1-p},$$

$$\left(\frac{3}{p}-1\right)^{2} \le \left(\frac{\frac{3}{p}-1}{\frac{1}{p}-1}\right)^{\frac{1}{p}}.$$

Denoting $t = \frac{1}{p} \in [3, \infty)$, we obtain the following more convenient equivalent form of the latter inequality.

$$(3t-1)^2 \le \left(\frac{3t-1}{t-1}\right)^t \iff t\ln\left(\frac{3t-1}{t-1}\right) \ge 2\ln(3t-1).$$

Let

$$h(t) = t\left[\ln(3t-1) - \ln(t-1)\right] - 2\ln(3t-1).$$

Then

$$\begin{aligned} h'(t) &= \ln(3t-1) - \ln(t-1) + t\left(\frac{3}{3t-1} - \frac{1}{t-1}\right) - \frac{6}{3t-1} \\ &= \ln(3t-1) - \ln(t-1) - \frac{1}{t-1} - \frac{5}{3t-1} \end{aligned}$$

and

$$h''(t) = \frac{3}{3t-1} - \frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{15}{(3t-1)^2} = \frac{2\left(9t^2 - 14t + 7\right)}{(3t-1)^2(t-1)^2}.$$

Since $h''(t) > 0$ for $t \ge 3$, $h'(t)$ increases on $[3, \infty)$ and, therefore,

$$h'(t) \ge h'(3) = \ln 8 - \ln 2 - \frac{1}{2} - \frac{5}{8} = 2\ln 2 - \frac{9}{8} > 0.$$

Hence, $h(t)$ increases on $[3, \infty)$ and, therefore,

$$h(t) \ge h(3) = 3(\ln 8 - \ln 2) - 2\ln 8 = 0.$$

Thus, $t\ln\left(\frac{3t-1}{t-1}\right) \ge 2\ln(3t-1)$, as desired.

4143. *Proposed by Roy Barbara.*

For any real number $x \ge 1$, let $y = x^{1/2} + x^{-1/2}$.

a) Express x in terms of y by a radical formula and check that no rational fraction $F(t)$ can exist such that $x = F(y)$. (A rational fraction is an expression of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials with rational coefficients.)

b) Find a closed form formula $x = F(y)$ containing no radicals.

c) $\star$ Is there a *complex* fraction such that $x = F(y)$? (A complex fraction is a function of the form $f(z)/g(z)$, where $f(t)$ and $g(t)$ are polynomials with complex coefficients.)

We received four solutions, all correct, and feature that of Joseph DiMuro.

a) We have $y = \frac{x+1}{\sqrt{x}}$, which can be rewritten as $x - y\sqrt{x} + 1 = 0$. The quadratic formula then gives us $\sqrt{x} = \frac{y \pm \sqrt{y^2-4}}{2}$. Both of these possible expressions for $\sqrt{x}$

are positive, and their product is 1, so exactly one of them is greater than 1. We are given that $\sqrt{x} \geq 1$, so we choose the larger solution; that is, $\sqrt{x} = \frac{y+\sqrt{y^2-4}}{2}$, and therefore,

$$x = \frac{1}{2}y^2 + \frac{1}{2}y\sqrt{y^2-4} - 1.$$

If, to the contrary, there were a rational fraction $F(t)$ such that $x = F(y)$, then x would be a rational number whenever y is a rational number. But when $y = 4$, we have $x = 7 + 4\sqrt{3}$, which is irrational. We conclude that no such rational fraction $F(t)$ exists.

b) Letting $\sec\theta = \frac{y}{2}$, we have $\tan\theta = \frac{\sqrt{y^2-4}}{2}$. Consequently,

$$\tan\left(\sec^{-1}\left(\frac{y}{2}\right)\right) = \frac{\sqrt{y^2-4}}{2},$$

which allows us to write $\frac{\sqrt{y^2-4}}{2}$ without any radicals, specifically

$$x = \frac{1}{2}y^2 + y\tan\left(\sec^{-1}\left(\frac{y}{2}\right)\right) - 1.$$

c) Assume that there is a complex fraction $F(y)$ such that $F(y) = \frac{1}{2}y^2 + \frac{1}{2}y\sqrt{y^2-4} - 1$. We then have

$$\sqrt{y^2-4} = \frac{F(y) - \frac{1}{2}y^2 + 1}{\frac{1}{2}y},$$

so $\sqrt{y^2-4}$ can itself be expressed as a complex fraction. Let

$$\sqrt{y^2-4} = \frac{f(y)}{g(y)},$$

where $f(y)$ and $g(y)$ are polynomials with complex coefficients. We may assume that the fraction is in lowest terms, so that $f(y)$ and $g(y)$ have no zeros in common. We then have

$$y^2 - 4 = \frac{(f(y))^2}{(g(y))^2}.$$

But this is impossible; all zeros of $\dfrac{(f(y))^2}{(g(y))^2}$ have even multiplicity, but the two zeros of y^2-4 (namely, ± 2) each have multiplicity 1. So *there can be no complex fraction $F(y)$ such that $x = F(y)$.*

Editor's Comments. The other formulas "containing no radicals" that we received were

$$x = \exp\left(2\cosh^{-1}\frac{y}{2}\right) \quad \text{and} \quad x = \cot^2\left(\frac{1}{2}\arcsin\frac{2}{y}\right).$$

These display, of course, simply cosmetic differences: radicals are historically important special cases of exponential functions.

4144. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=1$. Find the maximum value of the expression

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3.$$

We received 19 submissions, all of which are correct. We present two solutions with the first one being a composite of very similar solutions by several solvers.

Solution 1, by Arkady Alt, Michel Bataille, Steven Chow, and Daniel Dan (independently).

Let $q = ab+bc+ca$ and $r = abc$. Then

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1-2q$$

and

$$a^3+b^3+c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc = 1-3q+3r.$$

Hence,

$$\begin{aligned}&\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3\\ &= a^3+b^3+c^3-\frac{3}{2}(a^2+b^2+c^2)+\frac{3}{4}(a+b+c)-\frac{3}{8}\\ &= 1-3q+3r-\frac{3}{2}(1-2q)+\frac{3}{8} = 3r-\frac{1}{8} = 3abc-\frac{1}{8}\\ &\le 3\left(\frac{a+b+c}{3}\right)^3-\frac{1}{8} = \frac{1}{9}-\frac{1}{8} = -\frac{1}{72}.\end{aligned}$$

Therefore, the required maximum $-\frac{1}{72}$ is attained exactly when $a=b=c=\frac{1}{3}$.

Solution 2, by Titu Zvonaru.

We prove that the searched maximum is $-\frac{1}{72}$.

For convenience, let $d = a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$. Then

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3 \le -\frac{1}{72}$$

is equivalent in succession to

$$\begin{aligned}&9((2a-1)^3+(2b-1)^3+(2c-1)^3) \le 1,\\ &9((a-b-c)^3+(b-c-a)^3+(c-a-b)^3) \le -(a+b+c)^3,\\ &-9(a^3+b^3+c^3)-27d+162abc \le -(a^3+b^3+c^3)-3d-6abc,\\ &8(a^3+b^3+c^3)+24d \ge 168abc,\\ &a^3+b^3+c^3+3d \ge 21abc,\end{aligned}$$

which is true since by the AM-GM inequality we have

$$a^3+b^3+c^3+3d \geq 3abc+3(6abc)=21abc.$$

4145. *Proposed by Leonard Giugiuc.*

Prove that the system

$$\begin{cases} A^3+A^2B+AB^2+ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ B^3+B^2A+BA^2+BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$

has no solutions in the set of 3×3 matrices over complex numbers.

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Assume by contradiction that there exist matrices A and B that are solutions to this system.

$$\begin{cases} A^3+A^2B+AB^2+ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, & (1) \\ \\ B^3+B^2A+BA^2+BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. & (2) \end{cases}$$

Equation (1) gives

$$A(A+B)^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(A)\cdot\det(A+B)^2 = 1$$
$$\Rightarrow \ \det(A+B) \neq 0. \qquad (3)$$

From equations (1) and (2), we get

$$(A+B)^3 = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ \det(A+B)^3 = 0 \ \Rightarrow \ \det(A+B)=0,$$

which contradicts (3).

4146. *Proposed by Mehmet Berke Işler.*

Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 2(ab + bc + ca)$ and $\sqrt{a}+\sqrt{b}+\sqrt{c} = 2$. Prove that at least one of the numbers a, b, c is equal to 1.

There were seventeen correct solutions. Nine of them took the approach of Solution 1, two of Solution 2, two of Solution 3, and two of Solution 4. Below we present all four types of solutions by various solvers.

Solution 1.

Let $(a, b, c) = (x^2, y^2, z^2)$ with $x, y, z \geq 0$. Then

$$\begin{aligned} 0 &= 2(x^2y^2 + y^2z^2 + z^2x^2) - (x^4 + y^4 + z^4) \\ &= (x + y + z)(x + y - z)(y + z - x)(z + x - y) \\ &= 16(1 - z)(1 - x)(1 - y), \end{aligned}$$

from which at least one of x, y, z equals 1.

Solution 2.

The first condition can be rewritten as

$$\begin{aligned} 0 &= a^2 - 2(b + c)a + (b - c)^2 \\ &= [a - (\sqrt{b} - \sqrt{c})^2][a - (\sqrt{b} + \sqrt{c})^2]. \end{aligned}$$

This implies that $\sqrt{a} = \pm(\sqrt{b} - \sqrt{c})$ or $\sqrt{a} = \sqrt{b} + \sqrt{c}$, each of which implies that two of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ add to the third. Thus, in view of the second condition, we are led to one of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ equal to 1.

Solution 3.

Since $(b + c - a)^2 = 4bc$, then $|b + c - a| = 2\sqrt{bc}$, from which $(\sqrt{b} \pm \sqrt{c})^2 = (\sqrt{a})^2$. Thus, as in the previous solution, two of the square roots add to the third and we reach the desired conclusion.

Solution 4.

With $(a, b, c) = (x^2, y^2, z^2)$, let $s = x+y+z = 2$, $q = xy+yz+zx$, $p = xyz$. Then

$$\begin{aligned} 0 &= 2(x^2y^2 + y^2z^2 + z^2x^2) - (x^4 + y^4 + z^4) \\ &= 4(x^2y^2 + y^2z^2 + z^2x^2) - (x^2 + y^2 + z^2)^2 \\ &= 4(q^2 - 2ps) - (s^2 - 2q)^2 = 4s^2q - 8ps - s^4 \\ &= 16(q - p - 1), \end{aligned}$$

whence $q = 1 + p$. Since x, y, z satisfy the equation

$$0 = t^3 - st^2 + qt - p = t^3 - 2t^2 + t + p(t - 1) = (t - 1)(t^2 - t + p),$$

one of x, y, z equals 1.

Editor's Comments. Note that, in the notation of Solutions 1 and 4, when $z = 1$, the second condition becomes $x + y = 1$ and the first condition is automatically satisfied. If x, y, z are sides of a triangle, the conditions state that the triangle has perimeter 2 and area 0, so that it is degenerate and the length of the longest side is equal to the sum of the lengths of the other sides.

4147. *Proposed by Mehtaab Sawhney.*

Let $\{a_i\}$ be a sequence of real numbers. Suppose that $|a_i - a_j| \geq 2^{i-j}$ if $i > j$, then find the minimal value of

$$\sum_{1 \leq i < j \leq n} (a_j - a_i)^2.$$

The problem was solved by Steven Chow and the proposer. The following solution uses elements of both their solutions. An additional submission was incorrect.

We begin by showing that it suffices to consider only increasing sequences whose first term is 0. Let $\{a_1, \ldots, a_n\}$ be a sequence satisfying the conditions and let $\{b_1, \ldots, b_n\}$ be a reordering of its terms so that $b_1 < b_2 < \cdots < b_n$. Then $\sum(a_j - a_i)^2 = \sum(b_j - b_i)^2$ and $b_j - b_i \geq 2^{j-i}$ for $1 \leq i < j \leq n$.

The first conclusion is clear. As for the second, let $1 \leq i < j \leq n$. Consider the subsequence $B = \{b_i, b_{i+1}, \ldots, b_j\}$ with $j - i + 1$ entries. Let u be the minimum index and v be the maximum index for which a_u and a_v belong to B. Then $B \subseteq \{a_u, a_{u+1}, \ldots, a_v\}$, a set with $v - u + 1$ entries. Therefore $v - u \geq j - i$ and

$$b_j - b_i \geq |a_v - a_u| \geq 2^{|v-u|} \geq 2^{j-i}.$$

Since adding a constant to each term of a sequence does not alter either the given conditions or the square sum in the conclusion, we can, wolog, assume that $0 = a_1 < a_2 < \cdots < a_n$ so that $a_i \geq 2^{i-1}$ for $2 \leq i \leq n$.

Suppose now that $\{a_k\}$ is an increasing sequence with $a_1 = 0$ that minimizes the square sum. We show that $a_n = 2^{n-1}$.

Suppose, if possible, that $a_i > 2^{i-1}$ for $i \geq 2$. Let $m = \min\{a_i - 2^{i-1} : 2 \leq i \leq n\}$; note that $m > 0$. Suppose that $b_1 = a_1 = 0$ and $b_i = a_i - m$ for $2 \leq i \leq n$. Then $b_j < a_j$ and $b_j \geq a_j - (a_j - 2^{j-1}) = 2^{j-1}$ for $j \geq 2$; also $b_j - b_i = a_j - a_i$ for $j > i \geq 2$. Thus $\{b_i\}$ satisfies the conditions of the problem and the square sum is strictly smaller, contradicting the mimimality of $\{a_k\}$. Therefore, there is at least one value of k for which $a_k = 2^{k-1}$.

Let k be the largest index with $a_k = 2^{k-1}$. If $k < n$, suppose that $m = \min\{a_j - 2^{j-1} : k + 1 \leq i \leq n\}$. Define $b_i = a_i$ for $1 \leq i \leq k$ and $b_j = a_j - m$ for $k + 1 \leq j \leq m$. When $1 \leq i \leq k < j \leq n$,

$$\begin{aligned} b_j - b_i = b_j - b_k + (b_k - b_i) &\geq 2^{j-1} - 2^{k-1} + 2^{k-i} \\ &= 2^{j-i} + (2^{j-i} - 2^{k-i})(2^{i-1} - 1) \geq 2^{j-i}. \end{aligned}$$

It is now clear that $\{b_i\}$ satisfies the conditions of the problem and that the square sum is strictly smaller. This again contradicts minimality, and so $a_n = 2^{n-1}$.

The minimizing sequence exhibits a kind of bilateral symmetry. Let $\{a_k\}$ be such a sequence, with $x_k = a_{k+1} - a_k > 0$ for $1 \le k \le n-1$. Then for $1 \le i < j \le n$,

$$\begin{aligned}(a_j - a_i)^2 + (a_{n+1-i} - a_{n+1-j})^2 &\ge 2\left[\frac{(a_j - a_i) + (a_{n+1-i} - a_{n+1-j})}{2}\right]^2 \\ &= 2\left[\sum_{k=i}^{j-1} \frac{x_k + x_{n-k}}{2}\right]^2 = 2\left[\sum_{k=i}^{j-1} y_k\right]^2 \\ &= \left(\sum_{k=i}^{j-1} y_k\right)^2 + \left(\sum_{k=n-(j-1)}^{n-i} y_k\right)^2\end{aligned}$$

where $y_k = \frac{1}{2}(x_k + x_{n-k}) = y_{n-k}$ for $1 \le k \le n-1$.

Define $b_1 = a_1 = 0$, and $b_k = y_1 + y_2 + \cdots + y_{k-1}$ for $1 \le k \le n-1$. Then, for $1 \le i < j \le n$,

$$\begin{aligned}b_j - b_i = y_i + \cdots + y_{j-1} &= \frac{1}{2}\left[(x_i + \cdots + x_{j-1}) + (x_{n-j+1} + \cdots + x_{n-i})\right] \\ &= \frac{1}{2}\left[(a_j - a_i) + (a_{n+1-i} - a_{n-j+1})\right] \\ &\ge 2^{j-i},\end{aligned}$$

$b_n = a_n = 2^{n-1}$ and

$$\sum_{1\le i<j\le n} (b_j - b_i)^2 \le \sum_{1\le i<j\le n} (a_j - a_i)^2.$$

Since $\{a_k\}$ was minimal, we must have $a_k = b_k$ for each k.

We have a final step. We subtract from each term of our optimal sequence the number 2^{n-2} to get a *balanced* sequence $\{a_k : 1 \le k \le n\}$ which satisfies the conditions of the problem along with the condition that

$$a_k = -2^{n-2} + y_1 + y_2 + \cdots + y_{k-1} = -(2^{n-2} - y_1 - y_2 - \cdots - y_{k-1}) = -a_{n+1-k}$$

for $2 \le k \le n-1$; in particular $a_n = 2^{n-2} = -a_1$. Note in particular that $|2a_k| = |a_k - a_{n+1-k}| \ge 2^{n-2k}$ for $1 \le k \le n$.

It is now time to introduce the minimizing candidates. Let $k \ge 1$ and define the balanced sequences $A_n = \{a_1, \ldots, a_n\}$ by

$$A_{2m} = \{-4^{m-1}, -4^{m-2}, \ldots, -4^0 = -1, 4^0 = 1, \ldots, 4^{m-2}, 4^{m-1}\},$$

$$\begin{aligned}A_{2m+1} &= \{-2^{2m-1}, -2^{2m-3}, \ldots, -2, 0, 2, \ldots, 2^{2m-3}, 2^{2m-1}\} \\ &= 2\{-4^{m-1}, -4^{m-2}, \ldots, -1, 0, 1, \ldots, 4^{m-2}, 4^{m-1}\}.\end{aligned}$$

It is readily checked that each A_n has n terms and satisfies the conditions of the problem; in crucial circumstances it does so with equality, to wit

$$a_{m+k} - a_{m+1-k} = 2(4^{k-1}) = 2^{2k-1}$$

when $n = 2m$ and $1 \leq k \leq m$, and

$$a_{m+1+k} - a_{m-k} = 2(4^{k-1}) = 2^{2k-1}$$

when $n = 2m + 1$ and $1 \leq k \leq m$. We will evaluate the square sum for these sequences.

Let S_m be the sum $\sum_{1 \leq i < j \leq 2m} (a_j - a_i)^2$ for the sequence A_{2m}.

Noting that the sequence $A_{2(m+1)}$ consists of the sequence A_{2m} with additional terms $\pm 4^k$ appended at the ends, we see that, for $m \geq 1$,

$$\begin{aligned} S_{m+1} &= S_m + (4^m + 4^m)^2 + 2\sum_{k=0}^{m-1} \left[(4^m - 4^k)^2 + (4^m + 4^k)^2\right] \\ &= S_m + 4^{2m+1} + m \cdot 4^{2m+1} + 4\left(\frac{4^{2m} - 1}{15}\right) \\ &= S_m + m \cdot 4^{2m+1} + \frac{1}{15}\left(4^{2m+3} - 4\right). \end{aligned}$$

Hence

$$\begin{aligned} S_m &= S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots + (S_m - S_{m-1}) \\ &= 4 + 4^3[1 + 2 \cdot 16 + 3 \cdot 16^2 + \cdots + (m-1)16^{m-2}] \\ &\quad + \frac{4^5}{15}[1 + 16 + \cdots + 16^{m-2}] - \frac{4(m-1)}{15} \\ &= 4 + \frac{4^3}{15^2}[(m-1)16^m - m \cdot 16^{m-1} + 1 + (16^m - 16)] - \frac{4(m-1)}{15} \\ &= \frac{4m(16^m - 1)}{15}. \end{aligned}$$

Thus, when $n = 2m$ is even, the sum $\sum (a_j - a_i)^2$ is equal to

$$\frac{4m(16^m - 1)}{15} = \frac{2n(4^n - 1)}{15}.$$

When $n = 2m + 1$ is odd, then $\sum (a_j - a_i)^2$ is equal to

$$\begin{aligned} 4S_m + 4 \cdot 2(1 + 16 + \cdots + 16^{m-1}) &= \frac{1}{15}[16m(16^m - 1) + 8(16^m - 1)] \\ &= \frac{8}{15}[(2m+1)(4^{2m} - 1)] \\ &= \frac{8n}{15}(4^{n-1} - 1). \end{aligned}$$

We now establish minimality. Clearly, $\{-1, 1\}$ is an optimizing sequence with two entries. Suppose we have an optimizing sequence for $n = 2m > 2$, namely the balanced

$$A = \{-4^{m-1}, a_2, a_3, \ldots, a_{2m-1}, 4^{m-1}\}.$$

Denoting by S the sum of all the squares of differences that do not involve the first and last terms, we find that the sum of the squares of the differences equals

$$\begin{aligned}
&S + (2 \cdot 4^{m-1})^2 + 2 \sum_{k=2}^{2m-1} (4^{m-1} - a_k)^2 \\
&= S + 4^{2m-1} + 4(m-1)4^{2m-2} - \left(4^m \sum_{k=2}^{m} (a_k + a_{2m+1-k}) \right) + 2 \sum_{k=2}^{2m-1} a_k^2 \\
&= T + 2 \sum_{k=-(m-2)}^{m-1} a_{m+k}^2 \\
&\geq T + 2 \sum_{k=2}^{m} (4^k),
\end{aligned}$$

where T is a constant, independent of the balanced sequence. In fact, by inserting $A_{2(k-1)}$ between -4^{k-1} and 4^{k-1}, we get equality at the last stage of the display. So we can work our way up from A_2 to find that A_{2m} is an optimizing sequence for each positive integer m.

We can follow a similar argument to show that A_{2n+1} is also optimizing for each positive integer m. Thus the minimum value of the square sum is

$$\frac{2n(4^n - 1)}{15}, \quad \text{when } n \text{ is even,}$$
$$\frac{8n(4^{n-1} - 1)}{15}, \quad \text{when } n \text{ is odd.}$$

This can also be rendered as

$$\frac{2n}{15} \left(4^n + (-1)^n \frac{3}{2} - \frac{5}{2} \right)$$

for all n.

4148. *Proposed by Lorian Saceanu.*

For positive real numbers x, y and z, show that

$$\begin{aligned}
&\sqrt{xy(x+y)} + \sqrt{yz(y+z)} + \sqrt{xz(x+z)} \\
&\qquad \geq \sqrt{(x+y)(y+z)(z+x)} + (x+y+z)\sqrt{\frac{2xyz}{3(xy+yz+xz)}}.
\end{aligned}$$

We received four correct submissions, out of which we present the solution by Oliver Geupel.

We start with the following calculation:

$$\left(\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{zx(z+x)}\right)\cdot\left(\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\right)$$
$$=x\sqrt{y(x+y)(y+z)}+y\sqrt{x(x+y)(x+z)}+\sqrt{xyz}(x+y)$$
$$+\sqrt{xyz}(y+z)+y\sqrt{z(y+z)(x+z)}+z\sqrt{y(y+z)(x+y)}$$
$$+x\sqrt{z(z+x)(y+z)}+\sqrt{xyz}(x+z)+z\sqrt{x(x+y)(x+z)}$$
$$=\sqrt{y(x+y)(y+z)}(x+z)+\sqrt{x(x+y)(x+z)}(y+z)$$
$$+\sqrt{z(x+z)(y+z)}(x+y)+2\sqrt{xyz}(x+y+z)$$
$$=\sqrt{(x+y)(y+z)(z+x)}\left(\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\right)$$
$$+2(x+y+z)\sqrt{xyz}.$$

Rearranging, we get as a consequence that

$$\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{zx(z+x)}$$
$$=\sqrt{(x+y)(y+z)(z+x)}+(x+y+z)\frac{\sqrt{4xyz}}{\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}}.$$

By Jensen's inequality for the square root function, we have

$$\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\le 3\sqrt{\frac{x(y+z)+y(z+x)+z(x+y)}{3}}$$
$$=\sqrt{6(xy+yz+zx)}.$$

The desired result immediately follows. Note that from the properties of Jensen's inequality, the equality holds if and only if

$$x(y+z)=y(z+x)=z(x+y),$$

that is, if and only if $x=y=z$.

4149. *Proposed by Daniel Sitaru.*

Prove that if $[a,b]\subset\left[0,\frac{\pi}{4}\right]$ then:

$$3(a\tan b+b\tan a)\ge ab(6+a\tan a+b\tan b).$$

We received five submissions, all correct. We present the solution by Digby Smith.

We first prove that if x is a real number such that $0\le x\le 1$, then

$$(3-x^2)\tan x\ge 3x. \tag{1}$$

From the Maclaurin series expansion for $\tan x$, we have that

$$\tan x \geq x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7.$$

Hence,

$$\begin{aligned}
&(3-x^2)\tan x - 3x \\
&\geq (3-x^2)\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7\right) - 3x \\
&= \left(3x + x^3 + \frac{6}{15}x^5 + \frac{51}{315}x^7\right) - \left(x^3 + \frac{1}{3}x^5 + \frac{2}{15}x^7 + \frac{17}{315}x^9\right) - 3x \\
&= \frac{1}{15}x^5 + \frac{9}{315}x^7 - \frac{17}{315}x^9 \\
&= \frac{1}{315}x^5(21 + 9x^2 - 17x^4) \geq 0,
\end{aligned}$$

which establishes (1).

Applying (1) with $x = a$ and b, respectively, we then have

$$(3-a^2)\tan a \geq 3a \quad \text{and} \quad (3-b^2)\tan b \geq 3b.$$

Therefore,

$$a(3-b^2)\tan b + b(3-a^2)\tan a \geq 6ab,$$

from which the given inequality follows immediately.

Editor's comment. Roy Barbara also proved (1) first and then used calculus with some elaborate calculations to actually show that the given inequality holds for all $a, b \in [0, \pi/2)$.

4150. *Proposed by Leonard Giugiuc.*

Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\left(x_n^2 + 2x_n + \frac{32}{x_n^3}\right) = 12.$$

Show that $\lim_{n\to\infty} x_n$ exists and find its value.

We received twelve correct and complete submissions. We present two solutions.

Solution 1, by the AN-anduud Problem Solving Group.

The statement in the question is equivalent to:

$$\forall\varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \left|x_n^2 + 2x_n + \frac{32}{x_n^3} - 12\right| < \varepsilon.$$

We calculate

$$\begin{aligned}\varepsilon &> \left|x_n^2 + 2x_n + \frac{32}{x_n^3} - 12\right|\\ &= \left|\frac{(x_n-2)^2(x_n^3+6x_n^2+8x_n+8)}{x_n^3}\right|\\ &= (x_n-2)^2\left(1+\frac{6}{x_n}+\frac{8}{x_n^2}+\frac{8}{x_n^3}\right).\end{aligned}$$

Since the right factor is greater than 1, we conclude that for all $n \geq n_0$: $(x_n-2)^2 < \varepsilon$ or $|x_n - 2| < \sqrt{\varepsilon}$. Thus

$$\lim_{n\to\infty} x_n = 2.$$

Solution 2, by C.R. Pranesachar.

We will use the following observation: If $\langle a_n\rangle$ and $\langle b_n\rangle$ are two real sequences such that $\lim_{n\to\infty} a_n = 0$ and $\langle b_n\rangle$ is bounded then

$$\lim_{n\to\infty} a_n b_n = 0.$$

From the limit given in the question we conclude that $\langle x_n\rangle$ has to be a bounded sequence, and thus $\langle x_n^3\rangle$ is bounded as well. Using our observation we then get

$$\begin{aligned}0 &= \lim_{n\to\infty}\left(x_n^2+2x_n+\frac{32}{x_n^3}-12\right)x_n^3\\ &= \lim_{n\to\infty}\left(x_n^5+2x_n^4-12x_n^3+32\right)\\ &= \lim_{n\to\infty}(x_n-2)^2\left(x_n^3+6x_n^2+8x_n+8\right).\end{aligned}$$

Since $x_n > 0$, we have

$$0 < \frac{1}{x_n^3+6x_n^2+8x_n+8} < \frac{1}{8},$$

and so again by the above observation

$$\begin{aligned}0 &= \lim_{n\to\infty}\left((x_n-2)^2\left(x_n^3+6x_n^2+8x_n+8\right)\right)\cdot\frac{1}{x_n^3+6x_n^2+8x_n+8}\\ &= \lim_{n\to\infty}(x_n-2)^2,\end{aligned}$$

which yields $\lim_{n\to\infty} x_n = 2$.