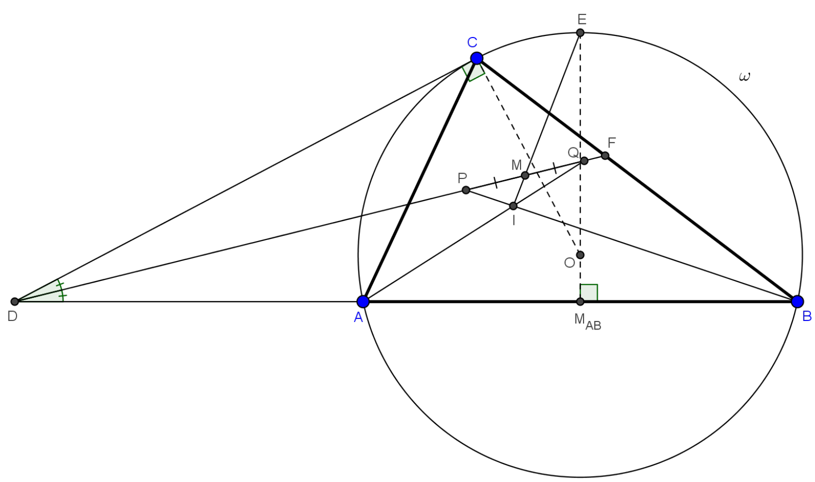
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(3), p. 102–103.

OC271. A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

Originally problem 7 of the Grade 11 2015 All Russian Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.



We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC.

The equation of the tangent to the circumcircle in C is given from

$$x\left(\frac{\partial f}{\partial x}\right)_C + y\left(\frac{\partial f}{\partial y}\right)_C + z\left(\frac{\partial f}{\partial z}\right)_C = 0,$$

where $f = a^2yz + b^2zx + c^2xy$ is the equation of the circumcircle, so we have

$$\left(\frac{\partial f}{\partial x}\right)_C = b^2, \quad \left(\frac{\partial f}{\partial y}\right)_C = a^2, \quad \left(\frac{\partial f}{\partial z}\right)_C = 0.$$

So the equation of the tangent to the circumcircle in C is $tg_C : b^2x + a^2y = 0$.

Point D has coordinates $D = AB \cap tg_C = (a^2 : -b^2 : 0)$.

We denote with F the point that is the intersection of the bisector of $\angle CDB$ with the side BC. To calculate its coordinates we remember that every bisector divides the opposite side in the ratio given from the lengths of the adjacent sides, therefore

$$\frac{BF}{FC} = \frac{BD}{CD}.$$

Now using the formula of distance between two points we have

$$BD = \frac{a^2c}{a^2 - b^2}, \qquad CD = \frac{abc}{a^2 - b^2},$$

from which we obtain

$$\frac{BF}{FC} = \frac{a}{b} \quad \Rightarrow \quad F(0 : b : a),$$

so the bisector of $\angle CDB$ has equation $DF : b^2x + a^2y - abz = 0$. The incenter I has coordinates $I(a : b : c)$, so the lines AI and BI have equations

$$AI : cy - bz = 0, \qquad BI : cx - az = 0.$$

Therefore, points P and Q have coordinates

$$P = BI \cap DF = \left(a^2 : b(c - b) : ac\right), \qquad Q = AI \cap DF = \left(a(c - a) : b^2 : bc\right).$$

The areal coordinates of points P and Q are

$$P = \left(\frac{a^2}{2(a + b)(s - b)}, \frac{b(c - b)}{2(a + b)(s - b)}, \frac{ac}{2(a + b)(s - b)}\right),$$

$$Q = \left(\frac{a(c - a)}{2(a + b)(s - a)}, \frac{b^2}{2(a + b)(s - a)}, \frac{bc}{2(a + b)(s - a)}\right),$$

so the coordinates of the midpoint of PQ are

$M\left(a(c^2 - 2a^2 + 2ab + ac - bc) : b(c^2 - 2b^2 + 2ab + bc - ac) : c(-a^2 - b^2 + 2ab + ac + bc)\right)$

and the equation of line MI is

$$MI : bc(s - a)x - ac(s - b)y + ab(a - b)z = 0.$$

Therefore, the intersection between MI and the circumcircle gives the coordinates of point E

$$E\left(a(b - a) : b(a - b) : c^2\right).$$

The line that passes from the midpoint M_{AB} of the side AB and is perpendicular to this side has equation

$$M_{AB}AB_{\infty\perp} : -c^2x + c^2y + (S_A - S_B)z = 0.$$

Now it is easy to verify that point E belongs to this line so we are done.

OC272. Find all real triples (a, b, c), for which

$$a(b^2 + c) = c(c + ab),$$
$$b(c^2 + a) = a(a + bc),$$
$$c(a^2 + b) = b(b + ca).$$

Originally problem 4 of the 2015 Czech and Slovak Olympiad III.

We received 2 correct submissions. We present the solution by Steven Chow.

The equations are equivalent to

$$ab(b - c) = c(c - a),$$
$$bc(c - a) = a(a - b),$$
$$ca(a - b) = b(b - c).$$

By multiplying these 3 equations, we get

$$ab(b - c)bc(c - a)ca(a - b) = c(c - a)a(a - b)b(b - c),$$

so either

$$a = 0 \iff b = 0 \iff c = 0 \text{ or } a = b \iff b = c \iff c = a \text{ or } abc = 1.$$

Therefore either $a = b = c$ which satisfies the system, or $abc = 1$.

If $abc = 1$, then the equations are equivalent to

$$b - c = c^2(c - a),$$
$$c - a = a^2(a - b),$$
$$a - b = b^2(b - c),$$

which are all non-negative or all non-positive. Adding these 3 equations,

$$0 = c^2(c - a) + a^2(a - b) + b^2(b - c).$$

Thus, each term on the right hand side must be equal to 0, so either

$$a = 0 \iff b = 0 \iff c = 0 \text{ or } a = b \iff b = c \iff c = a.$$

Therefore all real triples (a, b, c) are all $(a, b, c) \in \{(r, r, r) : r \in \mathfrak{Re}\}$.

OC273. Find all functions $f : R \to R$ such that $f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015}$ holds for all reals x, y.

Originally problem 1 from day 1 of the 2015 Final Round Korean Math Olympiad.

We received 2 correct submissions. We present the solution by Michel Bataille.

It can be checked that functions f_1 and f_2 defined by $f_1(x) = x$ and $f_2(x) = -x$ are solutions. We show that there are no other solutions. To this aim, we consider an arbitrary solution f and first show that $f(0) = 0$. We set $n = 2015$ for convenience and let $a = f(0)$. Taking $x = -a, y = 0$, the equation gives

$$a = (f(-a))^n \tag{1}$$

and, taking $x = 0, y = -a$, $f(f(-a))^n) = 0$. With (1), we deduce

$$f(a) = 0 \tag{2}$$

and with $x = 0, y = a$, we then obtain

$$a = 2a^n. \tag{3}$$

Now, we have

$$f\left(f(x^n + (f(y))^n)\right) = f(y^n + (f(x))^n) = (f(y))^n + x^n$$

for all x, y, a relation which, with $y = a$, provides $f \circ f(x^n) = x^n$. Since n is odd, the function $x \mapsto x^n$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$ and it follows that $f \circ f(u) = u$ for all real u. Note that this implies that f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$.

With $y = f(x)$, the equation gives

$$(f(x))^n = \frac{1}{2} \cdot f(2x^n).$$

Hence from (1), $a = (f(-a))^n = \frac{1}{2}f(-2a^n)$ and using (3), $f(-a) = 2a$. Finally, with (1), we arrive at

$$a = (f(-a))^n) = (2a)^n = 2^n a^n.$$

If we had $a \neq 0$, then the latter would give $a^{n-1} = \frac{1}{2^n}$ while (3) gives $a^{n-1} = \frac{1}{2}$, a contradiction. Thus $a = 0$, that is, $f(0) = 0$.

The equation now yields $f(x^n) = (f(x))^n$ for all x [with $y = 0$] and taking $y = f(z)$,

$$f(x^n + z^n) = (f(x))^n + (f(z))^n = f(x^n) + f(z^n)$$

for all x, z so that $f(u + v) = f(u) + f(v)$ for all u, v. It is well-known that this relation implies that f is odd and that $f(rx) = rf(x)$ for all real x and all rational numbers r. Let x be a real number with $x \neq 0$ and let $b = f(1)$. Then, for all rational r, we obtain on the one hand,

$$f((r + x)^n) = (f(r + x))^n = (rb + f(x))^n = \sum_{k=0}^{n} \binom{n}{k} r^k b^k (f(x))^{n-k}$$

and on the other hand

$$f((r + x)^n) = f\left(\sum_{k=0}^{n} \binom{n}{k} r^k x^{n-k}\right) = \sum_{k=0}^{n} \binom{n}{k} r^k f(x^{n-k}).$$

Thus, the polynomials

$$\sum_{k=0}^{n} \binom{n}{k} b^k (f(x))^{n-k} X^k \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} f(x^{n-k}) X^k$$

take the same value whenever X is a rational number. Therefore they must have the same coefficients. In particular, $nb^{n-1}f(x) = nf(x)$ so that $b^{n-1} = 1$ (note that $f(x) \neq 0$ since $x \neq 0$ and f is injective). Since $n-1$ is even, we deduce that $b = 1$ or $b = -1$. Also,

$$\binom{n}{2} b^{n-2}(f(x))^2 = \binom{n}{2} f(x^2).$$

If $b = 1$, then $f(x^2) = (f(x))^2 \geq 0$. It follows that $f(t) \geq 0$ if $t \geq 0$. If $0 \leq u < v$, then $f(v) - f(u) = f(v-u) \geq 0$, hence $f(v) \geq f(u)$. As a result, f is nondecreasing on $[0, \infty)$, hence on $\mathbb{R}$ (since f is odd). In conjunction with the equation $f(u+v) = f(u) + f(v)$, we classically obtain that f is a linear function and since $f(1) = 1$, $f = f_1$. In a similar way, $f = f_2$ if $f(1) = -1$ and we are done.

OC274. Find all triplets (x, y, p) of positive integers such that $\dfrac{xy^3}{x+y} = p$ where p is a prime number.

Originally problem 1 of the 2015 Greece National Olympiad.

We received 4 correct submissions. We present the solution by David Manes.

We will show that the only triplet (x, y, p) that satisfies the problem is $(14, 2, 7)$. Note that $\frac{14 \cdot 2^3}{14+2} = 7$.

Since all terms are positive, it follows that the fractional equation reduces to $p(x+y) = xy^3$. Therefore, either p divides x or p divides y since p is a prime. If p divides x, then $x = pr$ for some integer r. Then

$$\frac{xy^3}{x+y} = \frac{pry^3}{pr+y} = p$$

simplifies to $y = ry^3 - pr = r(y^3 - p)$ so that r is a divisor of y. Thus, $y = rt$ for some integer t so that

$$rt = r(r^3t^3 - p) \quad \text{or} \quad t(r^3t^2 - 1) = p.$$

Therefore, t is a divisor of prime p, whence either $t = 1$ or $t = p$. If $t = p$, then $y = rp = x$ in which case

$$p = \frac{x^4}{2x} = \frac{x^3}{2},$$

a contradiction. Hence $t = 1$ implies $y = r$ and

$$r^3 - 1 = p \quad \text{or} \quad p = (r-1)(r^2 + r + 1).$$

Therefore, $r-1=1$ and $r^2+r+1=p$. Hence, $r=2=y$, $p=7$ and $x=pr=14$.

On the other hand, assume towards a contradiction that p is a divisor of y. Then $y=pv$ for some integer v. Then

$$\frac{xy^3}{x+y}=\frac{xp^3v^3}{x+pv}=p \quad\Longrightarrow\quad x=pv(xpv^2-1)=y(xpv^2-1).$$

Therefore, y is a divisor of x so that $x=yt$ for some integer t. Then the fractional equation becomes

$$\frac{xy^3}{x+y}=\frac{y^4t}{y+yt}=p$$

which simplifies to $t(y^3-p)=p$. Therefore, t is a divisor of p, hence either $t=1$ or $t=p$. If $t=1$, then $p=\frac{y^3}{2}$, a contradiction. Therefore, $t=p$ so that

$$p=y^3-1=(y-1)(y^2+y+1).$$

For this case, $y-1=1$ and $y^2+y+1=p$. Hence, $y=2$ and $p=7$, a contradiction since p divides y. This contradiction proves that p cannot divide y.

OC275. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finishes piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l),(j,k),(j,l)$ for some $1 \leq i,j,k,l \leq n$, such that $i<j$ and $k<l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Originally problem 4 from day 2 of the 2015 USA Mathematical Olympiad.

We present the solution by Steven Chow. There were no other submissions.

Let A be the set of each different non-equivalent way Steve can pile the stones on the grid. Let B be the set of each way non-negative integers can correspond to each of the n columns and n rows such that the sum of the numbers for the columns and the sum of the numbers for the rows are each equal to m. From the Balls in Urns Formula, $|B|=\binom{m+n-1}{m}^2$.

Let $f:A\to B$ be the function such that for all $a\in A$, $f(a)$ is the element of B such that the number corresponding to each column or row of $f(a)$ is equal to the number of stones in that column or row of a.

Lemma 1. The function f is surjective.

Proof. We will use mathematical induction on m.

If $m = 1$, then it is trivially true.

Assume that for some integer $k \geq 1$, if $m = k$, then f is surjective.

Let $m = k + 1$. Let $y \in B$. Let column r be a column of y that corresponds to the greatest number among the columns. Let row s be a row of y that corresponds to the greatest number among the rows. Let y' be the result of y after 1 is subtracted from both the number corresponding to column r and the number corresponding to row s.

Now, let x' be a way Steve can pile the stones on the grid such that $f(x') = y'$ (which exists from the induction hypothesis). Let x be the result of x' after 1 stone is added to the square (r, s). Therefore $x \in A$ and $f(x) = y$, so f is surjective. □

Lemma 2. The function f is injective.

By mathematical induction on n. If $n = 1$, then it is trivially true.

Assume that for some integer $k \geq 1$, if $n = k$, then f is injective.

Let $n = k + 1$. Let $x, y \in A$ such that $f(x) = f(y)$.

Without loss of generality, assume that the number of stones at $(k + 1, k + 1)$ of x is less than or equal to that of y. Since $f(x) = f(y)$, therefore the number of stones in column $k + 1$ or row $k + 1$ of x is greater than or equal to that of y, so for certain integers $1 \leq j_1, j_2 \leq k$, there exist stone moves on $(j_1, k + 1)$ and $(k + 1, j_2)$ that can be performed such that the number of stones at $(k + 1, k + 1)$ of x is equal to that of y.

The numbers corresponding to the columns and rows are invariant under stone moves performed, so for certain integers $1 \leq j_1, j_2, j_3 \leq k$, there exist stone moves on $(j_1, k + 1)$ and (j_2, j_3), or $(k + 1, j_1)$ and (j_2, j_3) that can be performed such that for all integers $1 \leq i \leq k + 1$, the number of stones on $(i, k + 1)$ of x is equal to that of y, and the number of stones on $(k + 1, i)$ of x is equal to that of y.

From the induction hypothesis on the grid with squares (j_1, j_2) for all integers $1 \leq j_1, j_2 \leq k$, therefore $x = y$, so f is injective. □

Therefore f is bijective, so $|A| = |B| = \binom{m+n-1}{m}^2$. Therefore the number of different non-equivalent ways Steve can pile the stones on the grid is $\binom{m+n-1}{m}^2$.

. . . .

Congratulations to Steven Chow who went 5/5 on these Olympiad problems!