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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

It says May 2017 on the front page of this issue and it is May 2017 as I am writing this. I have not invented a time machine: ***Crux*** has cleared its backlog.

So here we are now, back to the present, ready for the future.

Kseniya Garaschuk

THE CONTEST CORNER

No. 55

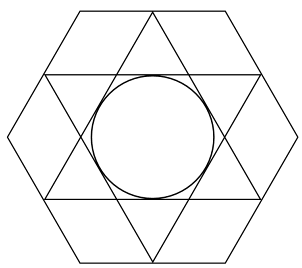
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **January 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC271. Warren's lampshade has an interesting design. Within a regular hexagon (six sides) are two intersecting equilateral triangles, and within them is a circle which just touches the sides of the triangles. (See the diagram.) The points of the triangles are at the midpoints of the sides of the hexagons.



If each side of the hexagon is 20 cm long, find:

a) the area of the hexagon;

b) the area of each large equilateral triangle;

c) the area of the circle.

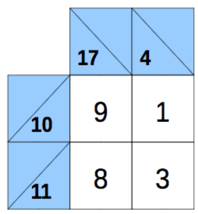
CC272. A *sum-palindrome number* (SPN) is a number that, when there are an even number of digits, the first half of the digits sums to the same total as the second half of the digits, and when odd, the digits to the left of the central digit sum to the same total as the digits to the right of the central digit. A *product-palindrome number* (PPN) is like a sum-palindrome, except the products of the digits are involved, not the sums.

a) How many three-digit SPNs are there?

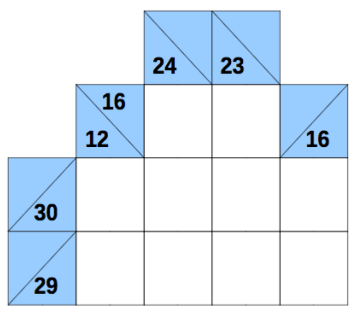
b) The two SPNs 1203 and 4022 sum to 5225, which is itself a SPN. Is it true that, for any two four-digit SPNs less than 5000, their sum is also a SPN?

c) How many four-digit non-zero PPNs are there?

CC273. Kakuro is the name of a number puzzle where you place numbers from 1 to 9 into empty boxes. There are three rules in a Kakuro puzzle: only numbers from 1 to 9 may be used, no number is allowed in any line (across or down) more than once, the numbers must add up to the totals shown at the top and the left. The diagram below shows a small finished Kakuro puzzle:



Solve the following Kakuro puzzle. Is your solution unique?



CC274. In his office, Shaquille had nine ping pong balls which he used for therapeutic recovery by throwing them into the waste basket at slack times. Each time he threw the nine balls, some of them would land in the basket, with the rest of them landing on the floor.

a) If the balls are identical, how many different results could there be?

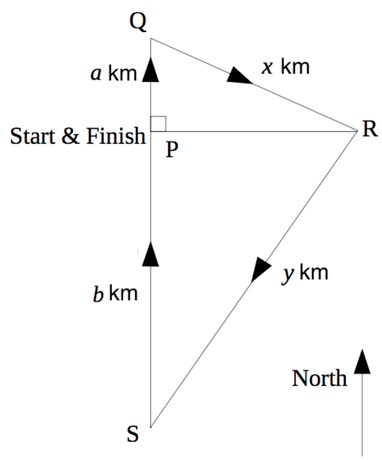
b) Suppose now that the balls are numbered 1 to 9. How many different results could there be now? (For example one possible result is for balls 1 to 4 to land in the basket, with 5 to 9 on the floor.)

c) Suppose instead that the balls are not numbered, but five are coloured yellow and four blue. Now how many different results could there be? (For example one possible result is for two yellow balls and three blue balls to land in the basket, and the rest to land on the floor.)

d) One day another basket appeared in the office. So now Shaquille had a choice of baskets to aim at. How did this change the answers to (a), (b), and (c)?

e) Now suppose that every time he threw the balls at the two baskets, each basket received at least two balls. How would this change the answers to (a), (b), and (c)?

CC275. The local sailing club is planning its annual race. By tradition the boats always start at P, sailing due North for a distance of a km until they reach Q. They then turn and sail a distance of x km to R (which is due East of P). Next they turn and sail a distance of y km to S (which is South of P) before finally sailing due North for a distance of b km until the finish line, which is back at the starting point P (see the diagram).



Bernie, the Club Commander, makes four extra rules for this year's race:

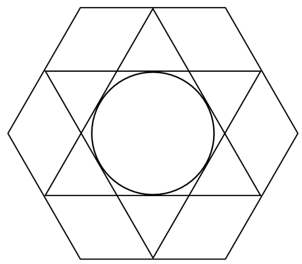
- $a + b = 40$,
- $x + y = 50$,
- $a < b$,
- the four lengths (a, b, x, y) must each be a whole number of kilometres. (Bernie doesn't like decimals.)

a) Find four numbers (a, b, x, y) which satisfy Bernie's four rules.

b) Are there four different numbers (a, b, x, y), which also satisfy Bernie's four rules apart from the four numbers you found in part (a)? Explain.

. . . .

CC271. Un abat-jour a un motif intéressant. À l'intérieur d'un hexagone régulier se trouvent deux triangles équilatéraux qui se coupent et à l'intérieur de ceux-ci, il y a un cercle qui touche aux côtés des triangles. (Voir la figure ce-dessous.) Les sommets des triangles sont les milieux des côtés de l'hexagone.



Sachant que chaque côté de l'hexagone a une longueur de 20 cm, déterminer:

a) l'aire de l'hexagone;

b) l'aire de chaque grand triangle équilatéral;

c) l'aire du cercle.

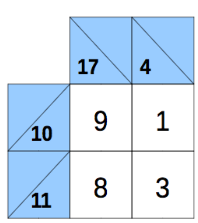
CC272. Un *nombre palindrome additif* (NPA) est un entier non négatif dont la somme de la première moitié de ses chiffres est égale à la somme de la seconde moitié de ses chiffres (s'il admet un nombre pair de chiffres) et s'il admet un nombre impair de chiffres, la somme des chiffres qui précèdent le chiffre central est égale à la somme des chiffres qui suivent le chiffre central. Un *nombre palindrome multiplicatif* (NPM) est défini de façon semblable, la somme étant remplacée par le produit.

a) Combien y a-t-il de NPA de trois chiffres?

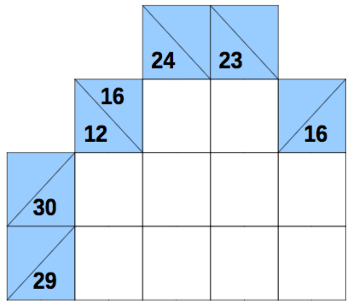
b) Le NPA 1203 et le NPA 4022 ont une somme de 5225, qui est un NPA. Est-il vrai qu'étant donné deux NPA de quatre chiffres, inférieurs à 5000, leur somme est toujours un NPA?

c) Combien y a-t-il de NPM de quatre chiffres?

CC273. Le Kakuro est un jeu logique comprenant une grille de cases dans lesquelles on place des chiffres de 1 à 9 pour former des nombres qui se lisent de gauche à droite ou de haut en bas. Le jeu comporte trois règles: seuls les chiffres de 1 à 9 peuvent être utilisés, un nombre ne peut contenir un même chiffre plus d'une fois, les chiffres d'un nombre doivent avoir une somme égale au nombre indiqué dans la case à la gauche ou au haut du nombre. La figure suivante montre un jeu Kakuro terminé:



Résoudre le jeu Kakuro suivant. La solution est-elle unique?



CC274. Dans son bureau, Shaquille s'amuse avec neuf balles de ping-pong qu'il tente de lancer dans une poubelle. Cela lui sert de thérapie par le jeu. Chaque fois qu'il lance les neuf balles, certaines d'entre elles tombent dans la poubelle, tandis que les autres se retrouvent par terre.

a) Sachant que les balles sont identiques, combien de résultats différents peut-il y avoir?

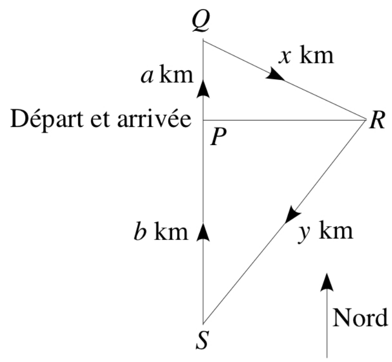
b) En supposant que les balles sont numérotées de 1 à 9, combien de résultats différents peut-il y avoir maintenant? (Par exemple, on peut considérer comme résultat possible que les balles de 1 à 4 tombent dans la poubelle et que les balles de 5 à 9 se retrouvent par terre.)

c) Supposons que les balles ne sont pas numérotées, mais que cinq d'entre elles sont jaunes et quatre sont bleues. Combien de résultats différents peut-il y avoir maintenant? (Par exemple, on peut considérer comme résultat possible que deux balles jaunes et trois balles bleues tombent dans la poubelle et que les autres se retrouvent par terre.)

d) Un bon jour, Shaquille hérite d'une deuxième poubelle pour son bureau. Il peut donc viser l'une ou l'autre poubelle. Comment cela change-t-il les réponses des parties (a), (b) et (c)?

e) Supposons qu'à chaque fois que Shaquille lance les neuf balles, chaque poubelle reçoit au moins deux balles. Comment cela change-t-il les réponses des parties (a), (b) et (c)?

CC275. Un club nautique prépare sa course d'hiver. Par tradition, les voiliers partent d'un point P et naviguent plein nord sur une distance de a km jusqu'au point Q. Ils changent de cap et parcourent x km en ligne droite jusqu'au point R situé à l'est de P. Ils changent de cap et parcourent y km en ligne droite jusqu'au point S situé au sud de P, pour ensuite naviguer plein nord sur une distance de b km jusqu'au point d'arrivée P. (Voir la figure ci-dessous.)



Bernard, le commandant du club nautique, ajoute quatre règlements:

- $a + b = 40$,
- $x + y = 50$,
- $a < b$,
- chacun des nombres a, b, x et y doit être un nombre entier strictement positif. (Bernard n'aime pas les nombres décimaux.)

a) Déterminer quatre nombres, a, b, x et y, qui vérifient les règlements de Bernard.

b) Y a-t-il quatre autres nombres, a, b, x et y, qui vérifient les règlements de Bernard autres que ceux déterminés dans la partie (a)? Expliquer.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 196–198.

CC221. What is the smallest positive integer n such that if S is any set containing n or more integers, then there must be three integers in S whose sum is divisible by 3?

Originally Question 29 from 2001 High School Math Contest of University of South Carolina.

We received twelve solutions, all of which were correct.

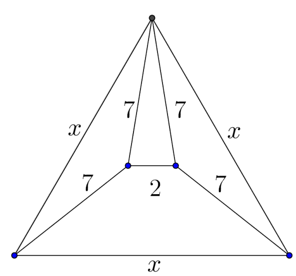
If there are three integers that have a sum divisible by three, then either

a) all three are congruent modulo 3, or

b) their residues modulo 3 must be 0, 1, and 2.

It is possible to avoid either of these scenarios in a set of three integers; for example, the elements in $\{3, 4, 6\}$ have residues (mod 3) of $\{0, 1, 0\}$ or in a set of four integers (take $\{3, 4, 6, 7\}$, with residues (mod 3) of $\{0, 1, 0, 1\}$). A set of five integers must, by the pigeonhole principle, either have three elements with the same residue (mod 3) (case (a)) or three elements with residues (mod 3) of 0, 1, and 2 (case (b)). Thus the smallest set is five.

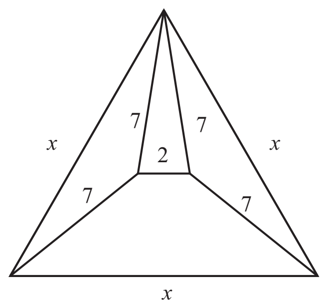
CC222. What is the value of x in the plane figure shown?

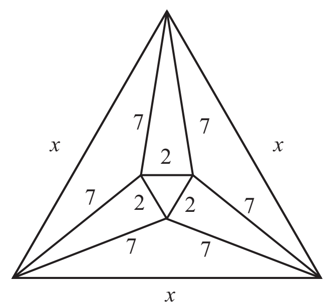


Originally Question 30 from 2002 High School Math Contest of University of South Carolina.

We received 14 correct and complete solutions, out of which we present the one by Ángel Plaza.

The figure below shows that by symmetry we can add other line segments and their known lengths.





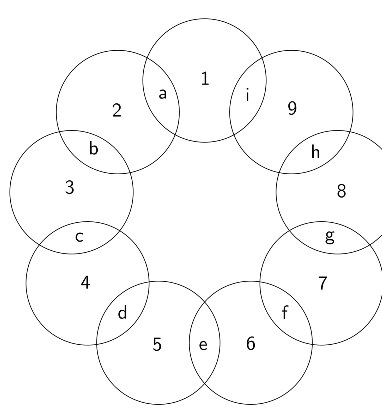
The area of the large equilateral triangle (in terms of x) may now be obtained in two ways, by directly calculating the area of the triangle, or by calculating the area of each of the sub-triangles. Thus

$$\frac{x^2\sqrt{3}}{2} = 3\left(\frac{x}{2}\sqrt{49-\frac{x^2}{4}}\right) + 3(\sqrt{48}) + \sqrt{3}.$$

The only positive solution of this equation is $x = 13$.

Editor's comments. For an alternative solution inspired by the symmetry of the diagram on the right, note that in an equilateral triangle with sidelength x the distance from a corner to the center is $\frac{x}{\sqrt{3}}$, two thirds of the height. As the center of the large equilateral triangle is the center of the inner equilateral triangle as well, we can also evaluate this distance as $\sqrt{48} + \frac{1}{\sqrt{3}} = \frac{13}{\sqrt{3}}$, which immediately gives $x = 13$.

CC223. The letters a, b, c, d, e, f, g, h and i in the figure below represent the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in a certain order. In each of the nine circles, we sum the three numbers so that nine sums are obtained. Suppose that all nine sums are equal. What is the value of $a + d + g$?



Originally Question 28 from 2003 High School Math Contest of University of South Carolina.

We received nine solutions, all of which were correct. We present an editor's amalgamation of the versions submitted by Somasundaram Muralidharan and Ricard Peiró i Estruch.

The sum of the numbers in the nine circles is

$$(1+2+\cdots+9)+2\cdot(a+b+\cdots+i)=3\cdot 45=135.$$

Since each circle contains the same sum, the sum in each circle must be $135/9=15$.

The circle holding the number 1 gives us $1+a+i=15$, so $a+i=14$. Since every letter must take on an integer value between 1 and 9 inclusive, the only possible pairs for (a,i) are $\{(6,8),(8,6),(5,9),(9,5)\}$. The circle to the right gives us $i+9+h=15$, so $i+h=6$, which implies $i\leq 5$. The only possibility for the pair (a,i) is $(9,5)$. Then $h=1$.

We can now continue around the circuit clockwise, filling in one new letter as we consider each new circle. The final answer is $a+d+g=9+3+6=18$.

CC224. What is the smallest positive integer n such that 31 divides 5^n+n?

Originally Question 29 from 2003 High School Math Contest of University of South Carolina.

We received 12 correct solutions and we present the solution by Steven Chow.

Observe that $5^1\equiv 5 \pmod{31}$, $5^2\equiv 25 \pmod{31}$, and $5^3\equiv 1 \pmod{31}$, so for all integers $a\geq 0$,

$$5^a\equiv\begin{cases}1 \pmod{31}, & \text{if } a\equiv 0 \pmod{3},\\ 5 \pmod{31}, & \text{if } a\equiv 1 \pmod{3},\\ 25 \pmod{31}, & \text{if } a\equiv 2 \pmod{3}.\end{cases}$$

If $n\equiv 0 \pmod 3$, then $0\equiv 5^n+n\equiv 1+n \pmod{31} \iff n\equiv 30 \pmod{31}$, so the least possible n is 30.

If $n\equiv 1 \pmod 3$, then $0\equiv 5^n+n\equiv 5+n \pmod{31} \iff n\equiv 26 \pmod{31}$, so the least possible n is 88.

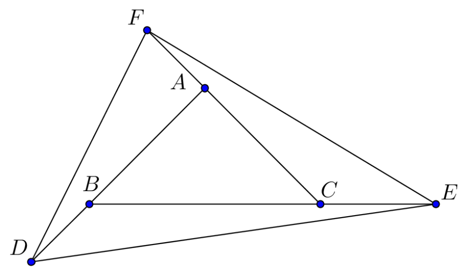
If $n\equiv 2 \pmod 3$, then $0\equiv 5^n+n\equiv 25+n \pmod{31} \iff n\equiv 6 \pmod{31}$, so the least possible n is 68.

Therefore the least possible value for n is 30.

Editor's Comments. David Manes used Fermat's Little Theorem in order to prove that $5^{30}+30\equiv 0 \pmod{31}$ and proved that there are no solutions for $n<30$. Konstantine Zelator showed also that the set of all positive integers n such that 5^n+n is divisible by 31 is the union of the three disjoint sets

$$\begin{aligned}S_1 &= \{n \mid n=93t+30, t\in\mathbb{N}\},\\ S_2 &= \{n \mid n=93q+88, q\in\mathbb{N}\},\\ S_3 &= \{n \mid n=93u+68, u\in\mathbb{N}\}.\end{aligned}$$

CC225. The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{2}AB, CE = \frac{1}{2}BC$ and $AF = \frac{1}{2}CA$. What is the ratio of the area of triangle DEF to that of triangle ABC?



Originally Question 30 from 2003 High School Math Contest of University of South Carolina.

We received eleven correct and two incorrect solutions. We present two solutions.

Solution 1, by Ricard Peiró i Estruch, slightly modified by the editor. The triangles ABC and AFB have the same altitude through A with the second triangle having half the base length of the first. Thus

$$[AFB] = \frac{1}{2}[ABC].$$

The triangles AFB and BFD have the same altitude through F, again with the second triangle having half the base length of the first, yielding

$$[BFD] = \frac{1}{2}[AFB] = \frac{1}{4}[ABC].$$

Similarly we obtain $[BDC] = [ACE] = \frac{1}{2}[ABC]$ and $[CDE] = [AEF] = \frac{1}{4}[ABC]$. Summing up all the areas gives

$$[DEF] = [ABC] + 3\left(\frac{1}{2}[ABC]\right) + 3\left(\frac{1}{4}[ABC]\right) = \frac{13}{4}[ABC].$$

Solution 2, by Andrea Fanchini. We use barycentric coordinates with respect to A, B, and C. Then the points D, E, F have coordinates

$$D\left(-\frac{1}{2}, \frac{3}{2}, 0\right), \qquad E\left(0, -\frac{1}{2}, \frac{3}{2}\right), \qquad F\left(\frac{3}{2}, 0, -\frac{1}{2}\right).$$

Therefore

$$[DEF] = \frac{[ABC]}{8}\begin{vmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \end{vmatrix},$$

implying

$$\frac{[DEF]}{[ABC]} = \frac{13}{4}.$$

THE OLYMPIAD CORNER

No. 353

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **January 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC331. Find all triples of nonnegative integers (x, y, z) and $x \leq y$ such that

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

OC332. Let $ABCD$ be a convex quadrilateral. Show that there exists a square $A'B'C'D'$ (where vertices may be ordered clockwise or counter-clockwise) such that $A \neq A', B \neq B', C \neq C', D \neq D'$ and AA', BB', CC', DD' are all concurrent.

OC333. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2.$$

OC334. Let p be an odd prime number. For positive integers k satisfying $1 \leq k \leq p - 1$, the number of divisors of $kp + 1$ between k and p exclusive is a_k. Find the value of $a_1 + a_2 + \cdots + a_{p-1}$.

OC335. Medians AM_A, BM_B and CM_C of a triangle ABC intersect at M. Let Ω_A be the circumcircle of the triangle that passes through the midpoint of AM and is tangent to BC at M_A. Define Ω_B and Ω_C analogously. Prove that Ω_A, Ω_B and Ω_C intersect at one point.

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OC331. Déterminer tous les triplets (x, y, z) d'entiers non négatifs $(x \leq y)$ tels que

$$x^2 + y^2 = 3 \cdot 2016^z + 77.$$

OC332. Soit $ABCD$ un quadrilatère convexe impair. Démontrer qu'il existe un carré $A'B'C'D'$ (dont les sommets peuvent être nommés dans le sens des aiguilles d'une montre ou dans le sens contraire) de manière que $A \neq A'$, $B \neq B'$, $C \neq C'$, $D \neq D'$ et que les droites AA', BB', CC' et DD' soient concourantes.

OC333. Déterminer toutes les fonctions f ($f : \mathbb{R} \to \mathbb{R}$) telles que

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2$$

pour tous réels x et y.

OC334. Soit p un nombre premier. Pour tout entier k $(1 \leq k \leq p - 1)$, soit a_k le nombre de diviseurs de $kp + 1$ situés entre k et p. Déterminer la valeur de $a_1 + a_2 + \cdots + a_{p-1}$.

OC335. Soit M le point d'intersection des médianes AM_A, BM_B et CM_C du triangle ABC. Soit Ω_A le cercle circonscrit au triangle qui passe au milieu de AM et qui est tangent à BC au point M_A. Ω_B et Ω_C sont définis de façon semblable. Démontrer que Ω_A, Ω_B et Ω_C sont concourantes.

Math Quotes

Like the ski resort full of girls hunting for husbands and husbands hunting for girls, the situation is not as symmetrical as it might seem.

Alan Lindsay Mackay in "A Dictionary of Scientific Quotations", Bristol: IOP Publishing, 1991.

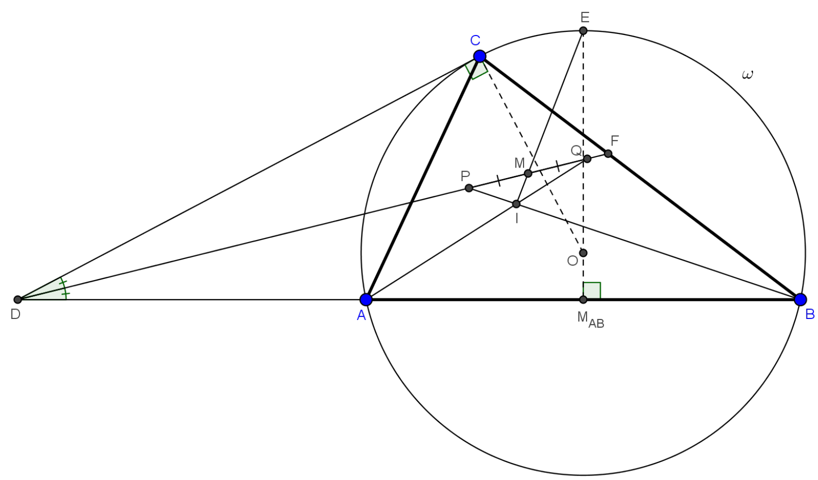
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(3), p. 102–103.

OC271. A scalene triangle ABC is inscribed within circle ω. The tangent to the circle at point C intersects line AB at point D. Let I be the center of the circle inscribed within $\triangle ABC$. Lines AI and BI intersect the bisector of $\angle CDB$ in points Q and P, respectively. Let M be the midpoint of QP. Prove that MI passes through the middle of arc ACB of circle ω.

Originally problem 7 of the Grade 11 2015 All Russian Olympiad.

We received 2 correct submissions. We present the solution by Andrea Fanchini.



We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC.

The equation of the tangent to the circumcircle in C is given from

$$x\left(\frac{\partial f}{\partial x}\right)_C + y\left(\frac{\partial f}{\partial y}\right)_C + z\left(\frac{\partial f}{\partial z}\right)_C = 0,$$

where $f = a^2yz + b^2zx + c^2xy$ is the equation of the circumcircle, so we have

$$\left(\frac{\partial f}{\partial x}\right)_C = b^2, \quad \left(\frac{\partial f}{\partial y}\right)_C = a^2, \quad \left(\frac{\partial f}{\partial z}\right)_C = 0.$$

So the equation of the tangent to the circumcircle in C is $tg_C : b^2x + a^2y = 0$.

Point D has coordinates $D = AB \cap tg_C = (a^2 : -b^2 : 0)$.

We denote with F the point that is the intersection of the bisector of $\angle CDB$ with the side BC. To calculate its coordinates we remember that every bisector divides the opposite side in the ratio given from the lengths of the adjacent sides, therefore

$$\frac{BF}{FC} = \frac{BD}{CD}.$$

Now using the formula of distance between two points we have

$$BD = \frac{a^2 c}{a^2 - b^2}, \qquad CD = \frac{abc}{a^2 - b^2},$$

from which we obtain

$$\frac{BF}{FC} = \frac{a}{b} \quad \Rightarrow \quad F(0 : b : a),$$

so the bisector of $\angle CDB$ has equation $DF : b^2x + a^2y - abz = 0$. The incenter I has coordinates $I(a : b : c)$, so the lines AI and BI have equations

$$AI : cy - bz = 0, \qquad BI : cx - az = 0.$$

Therefore, points P and Q have coordinates

$$P = BI \cap DF = \left(a^2 : b(c-b) : ac\right), \qquad Q = AI \cap DF = \left(a(c-a) : b^2 : bc\right).$$

The areal coordinates of points P and Q are

$$P = \left(\frac{a^2}{2(a+b)(s-b)}, \frac{b(c-b)}{2(a+b)(s-b)}, \frac{ac}{2(a+b)(s-b)}\right),$$

$$Q = \left(\frac{a(c-a)}{2(a+b)(s-a)}, \frac{b^2}{2(a+b)(s-a)}, \frac{bc}{2(a+b)(s-a)}\right),$$

so the coordinates of the midpoint of PQ are

$$M\left(a(c^2 - 2a^2 + 2ab + ac - bc) : b(c^2 - 2b^2 + 2ab + bc - ac) : c(-a^2 - b^2 + 2ab + ac + bc)\right)$$

and the equation of line MI is

$$MI : bc(s-a)x - ac(s-b)y + ab(a-b)z = 0.$$

Therefore, the intersection between MI and the circumcircle gives the coordinates of point E

$$E\left(a(b-a) : b(a-b) : c^2\right).$$

The line that passes from the midpoint M_{AB} of the side AB and is perpendicular to this side has equation

$$M_{AB}AB_{\infty\perp} : -c^2x + c^2y + (S_A - S_B)z = 0.$$

Now it is easy to verify that point E belongs to this line so we are done.

OC272. Find all real triples (a, b, c), for which

$$a(b^2 + c) = c(c + ab),$$
$$b(c^2 + a) = a(a + bc),$$
$$c(a^2 + b) = b(b + ca).$$

Originally problem 4 of the 2015 Czech and Slovak Olympiad III.

We received 2 correct submissions. We present the solution by Steven Chow.

The equations are equivalent to

$$ab(b - c) = c(c - a),$$
$$bc(c - a) = a(a - b),$$
$$ca(a - b) = b(b - c).$$

By multiplying these 3 equations, we get

$$ab(b - c)bc(c - a)ca(a - b) = c(c - a)a(a - b)b(b - c),$$

so either

$$a = 0 \iff b = 0 \iff c = 0 \text{ or } a = b \iff b = c \iff c = a \text{ or } abc = 1.$$

Therefore either $a = b = c$ which satisfies the system, or $abc = 1$.

If $abc = 1$, then the equations are equivalent to

$$b - c = c^2(c - a),$$
$$c - a = a^2(a - b),$$
$$a - b = b^2(b - c),$$

which are all non-negative or all non-positive. Adding these 3 equations,

$$0 = c^2(c - a) + a^2(a - b) + b^2(b - c).$$

Thus, each term on the right hand side must be equal to 0, so either

$$a = 0 \iff b = 0 \iff c = 0 \text{ or } a = b \iff b = c \iff c = a.$$

Therefore all real triples (a, b, c) are all $(a, b, c) \in \{(r, r, r) : r \in \mathfrak{Re}\}$.

OC273. Find all functions $f : R \to R$ such that $f(x^{2015} + (f(y))^{2015}) = (f(x))^{2015} + y^{2015}$ holds for all reals x, y.

Originally problem 1 from day 1 of the 2015 Final Round Korean Math Olympiad.

We received 2 correct submissions. We present the solution by Michel Bataille.

It can be checked that functions f_1 and f_2 defined by $f_1(x) = x$ and $f_2(x) = -x$ are solutions. We show that there are no other solutions. To this aim, we consider an arbitrary solution f and first show that $f(0) = 0$. We set $n = 2015$ for convenience and let $a = f(0)$. Taking $x = -a, y = 0$, the equation gives

$$a = (f(-a))^n \tag{1}$$

and, taking $x = 0, y = -a$, $f(f(-a))^n) = 0$. With (1), we deduce

$$f(a) = 0 \tag{2}$$

and with $x = 0, y = a$, we then obtain

$$a = 2a^n. \tag{3}$$

Now, we have

$$f\left(f(x^n + (f(y))^n)\right) = f(y^n + (f(x))^n) = (f(y))^n + x^n$$

for all x, y, a relation which, with $y = a$, provides $f \circ f(x^n) = x^n$. Since n is odd, the function $x \mapsto x^n$ is a bijection from $\mathbb{R}$ onto $\mathbb{R}$ and it follows that $f \circ f(u) = u$ for all real u. Note that this implies that f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$.

With $y = f(x)$, the equation gives

$$(f(x))^n = \frac{1}{2} \cdot f(2x^n).$$

Hence from (1), $a = (f(-a))^n = \frac{1}{2}f(-2a^n)$ and using (3), $f(-a) = 2a$. Finally, with (1), we arrive at

$$a = (f(-a))^n) = (2a)^n = 2^n a^n.$$

If we had $a \neq 0$, then the latter would give $a^{n-1} = \frac{1}{2^n}$ while (3) gives $a^{n-1} = \frac{1}{2}$, a contradiction. Thus $a = 0$, that is, $f(0) = 0$.

The equation now yields $f(x^n) = (f(x))^n$ for all x [with $y = 0$] and taking $y = f(z)$,

$$f(x^n + z^n) = (f(x))^n + (f(z))^n = f(x^n) + f(z^n)$$

for all x, z so that $f(u + v) = f(u) + f(v)$ for all u, v. It is well-known that this relation implies that f is odd and that $f(rx) = rf(x)$ for all real x and all rational numbers r. Let x be a real number with $x \neq 0$ and let $b = f(1)$. Then, for all rational r, we obtain on the one hand,

$$f((r + x)^n) = (f(r + x))^n = (rb + f(x))^n = \sum_{k=0}^{n} \binom{n}{k} r^k b^k (f(x))^{n-k}$$

and on the other hand

$$f((r + x)^n) = f\left(\sum_{k=0}^{n} \binom{n}{k} r^k x^{n-k}\right) = \sum_{k=0}^{n} \binom{n}{k} r^k f(x^{n-k}).$$

Thus, the polynomials

$$\sum_{k=0}^{n} \binom{n}{k} b^k (f(x))^{n-k} X^k \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} f(x^{n-k}) X^k$$

take the same value whenever X is a rational number. Therefore they must have the same coefficients. In particular, $nb^{n-1}f(x) = nf(x)$ so that $b^{n-1} = 1$ (note that $f(x) \neq 0$ since $x \neq 0$ and f is injective). Since $n-1$ is even, we deduce that $b = 1$ or $b = -1$. Also,

$$\binom{n}{2} b^{n-2} (f(x))^2 = \binom{n}{2} f(x^2).$$

If $b = 1$, then $f(x^2) = (f(x))^2 \geq 0$. It follows that $f(t) \geq 0$ if $t \geq 0$. If $0 \leq u < v$, then $f(v) - f(u) = f(v-u) \geq 0$, hence $f(v) \geq f(u)$. As a result, f is nondecreasing on $[0, \infty)$, hence on $\mathbb{R}$ (since f is odd). In conjunction with the equation $f(u+v) = f(u) + f(v)$, we classically obtain that f is a linear function and since $f(1) = 1$, $f = f_1$. In a similar way, $f = f_2$ if $f(1) = -1$ and we are done.

OC274. Find all triplets (x, y, p) of positive integers such that $\dfrac{xy^3}{x+y} = p$ where p is a prime number.

Originally problem 1 of the 2015 Greece National Olympiad.

We received 4 correct submissions. We present the solution by David Manes.

We will show that the only triplet (x, y, p) that satisfies the problem is $(14, 2, 7)$. Note that $\frac{14 \cdot 2^3}{14+2} = 7$.

Since all terms are positive, it follows that the fractional equation reduces to $p(x+y) = xy^3$. Therefore, either p divides x or p divides y since p is a prime. If p divides x, then $x = pr$ for some integer r. Then

$$\frac{xy^3}{x+y} = \frac{pry^3}{pr+y} = p$$

simplifies to $y = ry^3 - pr = r(y^3 - p)$ so that r is a divisor of y. Thus, $y = rt$ for some integer t so that

$$rt = r(r^3t^3 - p) \quad \text{or} \quad t(r^3t^2 - 1) = p.$$

Therefore, t is a divisor of prime p, whence either $t = 1$ or $t = p$. If $t = p$, then $y = rp = x$ in which case

$$p = \frac{x^4}{2x} = \frac{x^3}{2},$$

a contradiction. Hence $t = 1$ implies $y = r$ and

$$r^3 - 1 = p \quad \text{or} \quad p = (r-1)(r^2 + r + 1).$$

Therefore, $r-1=1$ and $r^2+r+1=p$. Hence, $r=2=y$, $p=7$ and $x=pr=14$.

On the other hand, assume towards a contradiction that p is a divisor of y. Then $y=pv$ for some integer v. Then

$$\frac{xy^3}{x+y}=\frac{xp^3v^3}{x+pv}=p \quad\implies\quad x=pv(xpv^2-1)=y(xpv^2-1).$$

Therefore, y is a divisor of x so that $x=yt$ for some integer t. Then the fractional equation becomes

$$\frac{xy^3}{x+y}=\frac{y^4t}{y+yt}=p$$

which simplifies to $t(y^3-p)=p$. Therefore, t is a divisor of p, hence either $t=1$ or $t=p$. If $t=1$, then $p=\frac{y^3}{2}$, a contradiction. Therefore, $t=p$ so that

$$p=y^3-1=(y-1)(y^2+y+1).$$

For this case, $y-1=1$ and $y^2+y+1=p$. Hence, $y=2$ and $p=7$, a contradiction since p divides y. This contradiction proves that p cannot divide y.

OC275. Steve is piling $m \geq 1$ indistinguishable stones on the squares of an $n \times n$ grid. Each square can have an arbitrarily high pile of stones. After he finishes piling his stones in some manner, he can then perform stone moves, defined as follows. Consider any four grid squares, which are corners of a rectangle, i.e. in positions $(i,k),(i,l),(j,k),(j,l)$ for some $1 \leq i,j,k,l \leq n$, such that $i<j$ and $k<l$. A stone move consists of either removing one stone from each of (i,k) and (j,l) and moving them to (i,l) and (j,k) respectively, or removing one stone from each of (i,l) and (j,k) and moving them to (i,k) and (j,l) respectively.

Two ways of piling the stones are equivalent if they can be obtained from one another by a sequence of stone moves.

How many different non-equivalent ways can Steve pile the stones on the grid?

Originally problem 4 from day 2 of the 2015 USA Mathematical Olympiad.

We present the solution by Steven Chow. There were no other submissions.

Let A be the set of each different non-equivalent way Steve can pile the stones on the grid. Let B be the set of each way non-negative integers can correspond to each of the n columns and n rows such that the sum of the numbers for the columns and the sum of the numbers for the rows are each equal to m. From the Balls in Urns Formula, $|B|=\binom{m+n-1}{m}^2$.

Let $f: A \to B$ be the function such that for all $a \in A$, $f(a)$ is the element of B such that the number corresponding to each column or row of $f(a)$ is equal to the number of stones in that column or row of a.

Lemma 1. The function f is surjective.

Proof. We will use mathematical induction on m.

If $m = 1$, then it is trivially true.

Assume that for some integer $k \geq 1$, if $m = k$, then f is surjective.

Let $m = k + 1$. Let $y \in B$. Let column r be a column of y that corresponds to the greatest number among the columns. Let row s be a row of y that corresponds to the greatest number among the rows. Let y' be the result of y after 1 is subtracted from both the number corresponding to column r and the number corresponding to row s.

Now, let x' be a way Steve can pile the stones on the grid such that $f(x') = y'$ (which exists from the induction hypothesis). Let x be the result of x' after 1 stone is added to the square (r, s). Therefore $x \in A$ and $f(x) = y$, so f is surjective. $\square$

Lemma 2. The function f is injective.

By mathematical induction on n. If $n = 1$, then it is trivially true.

Assume that for some integer $k \geq 1$, if $n = k$, then f is injective.

Let $n = k + 1$. Let $x, y \in A$ such that $f(x) = f(y)$.

Without loss of generality, assume that the number of stones at $(k + 1, k + 1)$ of x is less than or equal to that of y. Since $f(x) = f(y)$, therefore the number of stones in column $k + 1$ or row $k + 1$ of x is greater than or equal to that of y, so for certain integers $1 \leq j_1, j_2 \leq k$, there exist stone moves on $(j_1, k + 1)$ and $(k + 1, j_2)$ that can be performed such that the number of stones at $(k + 1, k + 1)$ of x is equal to that of y.

The numbers corresponding to the columns and rows are invariant under stone moves performed, so for certain integers $1 \leq j_1, j_2, j_3 \leq k$, there exist stone moves on $(j_1, k + 1)$ and (j_2, j_3), or $(k + 1, j_1)$ and (j_2, j_3) that can be performed such that for all integers $1 \leq i \leq k + 1$, the number of stones on $(i, k + 1)$ of x is equal to that of y, and the number of stones on $(k + 1, i)$ of x is equal to that of y.

From the induction hypothesis on the grid with squares (j_1, j_2) for all integers $1 \leq j_1, j_2 \leq k$, therefore $x = y$, so f is injective. $\square$

Therefore f is bijective, so $|A| = |B| = \binom{m+n-1}{m}^2$. Therefore the number of different non-equivalent ways Steve can pile the stones on the grid is $\binom{m+n-1}{m}^2$.

. . . .

Congratulations to Steven Chow who went 5/5 on these Olympiad problems!

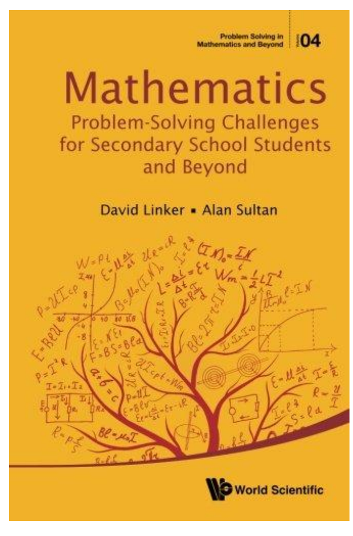
BOOK REVIEWS

Robert Bilinski

Mathematics: Problem-Solving Challenges for Secondary School Students and Beyond by D. Linker and A. Sultan
ISBN 9-789814-730037
World Scientific Press, 2016, 182 pages

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

The world abounds in marvellous endeavours to promote mathematics, but few get a chance at visibility outside of their niches. Luckily for us, once in a while, some contest organizers compile and publish a compendium book of problems. It takes dedication, hard work and perseverance to riffle through past contests, choose the cream of the crop, organize the problems, complete them with solutions, and polish the whole thing off. The New York City Interscholastic Math League (NYCIML) has been around for 33 years and last year they published such a book.



The NYCIML started in the fall of 1984 as a contest for high schools in the New York area. It has since grown to include teams from other parts of the United States, and now more than 100 teams participate each year. The contest has always had 4 categories: senior A and B, junior and frosh. Each year, students playing in the league have a few contests to write, depending on which grade they are enrolled in. An interesting aspect of the league's organization is that past participants become part of its executive committee.

The book's authors have different backgrounds. David Linker is on the contest and problem side of things; he participated in contests as a student and then coached teams as a mathematics teacher. He later became the head of the NYCIML and led it for a period of 11 years, all the while building a career at the City College of New York. Alan Sultan is a professor of mathematics at the City University of New York. He published 5 books and about 40 articles, specializing more on knowledge transfer and permanent education of high school mathematics teachers.

The book is divided into 9 chapters and includes an appendix. The chapters follow a standard structure, each collecting problems from the NYCIML covering different branches of mathematics, namely (in order): arithmetic and logic, algebra, geometry, trigonometry, logarithms, counting, number theory, probability and, lastly, functional equations. We can also see a progression in the complexity of the mathematical content. Each chapter contains anywhere from 33 to 162 problems, with most of them having around 50. The problems in each chapter are organized in three levels, from simpler to harder. But the complexity does not necessarily come from more complicated mathematics: some of the level 2 and 3 problems are not mathematically hard, instead featuring some kind of textual subtlety or word play. Level 3 problems are also generally more abstractly posed, while level 1 problems seem to be more "concrete" situations. Here are some examples.

Problem 1 of Chapter 2 (level 1). Find the largest integer x such that $\frac{24}{x+3}$ is an integer.

Problem 61 of Chapter 2 (level 2). If a and b are positive integers such that $a^2 + 24 = b^2$, compute the largest possible value of $a + b$.

Problem 153 of Chapter 2 (level 3). Find four ordered pairs of integers (x, y) such that $x^3 = y^3 + 217$.

The appendix of the book consists of an enumeration of mathematical results in the fields of each of the chapters. Contrary to most problems books, the authors opted for an answer section in each chapter instead of having a single solution section at the end of the book.

I appreciated that quite a few problems had multiple solutions. This is a definite plus. The problems also have a huge diversity, with only few problems resembling each other. Another interesting point is the number of problems touching different branches of math at the same time. For example, the authors put a counting problem as the first of the geometry section because of its geometric context (the problem asks how many diagonals are there in a given polygon). I do not know whether it was done on purpose, but the first problem of the counting section is also a geometry problem (namely, how many intersection points can two functions have). Even though I am an avid problem solver, I saw quite a few problems with a new twist or that I have never seen before: take, for example, problem 21 of the probability chapter that mixes probabilities with a Boolean implication.

It would have been nice to include references to the contest year and level from which each problem was taken. As someone who has been in the problem solving world for over 30 years and an owner of quite a collection of problem solving books, I figured I could date some problems from the way they were written, but I did not check my guesses.

A book of this kind can be used by teachers and math club organizers to spice up their presentations, while Mathletes would find it useful as a training tool to help them prepare for competitions. Maybe you know someone who is preparing for a math exam? This book is definitely a good starting point for that math teenage whiz in your family. If you know someone interested in keeping their mind sharp and who likes problem solving, this book would make a good gift for them.

Good reading!

FOCUS ON...

No. 26

Michel Bataille

Degree and roots of a polynomial

Introduction

In this number, we continue to illustrate some basic results about polynomials, focusing on the links between the degree and the number of roots.

Let $p(x)$ be a polynomial with coefficients in a field F. Recall that if k distinct elements $\alpha_1, \ldots, \alpha_k$ of F satisfy

$$p(\alpha_1) = \cdots = p(\alpha_k) = 0,$$

then $p(x)$ is divisible by

$$(x - \alpha_1) \cdots (x - \alpha_k).$$

A direct consequence is the following: unless $p(x)$ is the zero polynomial, we have $\deg(p(x)) \geq k$ and if $\deg(p(x)) = k$, then $p(x) = \rho(x - \alpha_1) \cdots (x - \alpha_k)$ for some nonzero constant ρ. In particular, a monic polynomial of degree k is completely determined once k distinct roots are identified.

Finding a polynomial through its roots

As an application of the latter, we consider the following nice problem set at the 1998 Vietnamese Olympiad:

> Prove that for each positive odd integer n there is exactly one polynomial $P(x)$ of degree n with real coefficients satisfying
>
> $$P\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n} \tag{1}$$
>
> for all real $x \neq 0$.

For a proof by induction, we refer the reader to [2003 : 456]. We propose a solution which, if slightly longer, has the advantage of giving an explicit expression of $P(x)$.

First, suppose that such a polynomial $P(x)$ exists and let us determine its complex roots. If w is any n-th root of unity in $\mathbb{C}$, then (1) gives $P\left(w - \frac{1}{w}\right) = 0$. It follows that the n complex numbers

$$\exp\left(\frac{2k\pi i}{n}\right) - \exp\left(\frac{-2k\pi i}{n}\right) = 2i \sin\frac{2k\pi}{n}, \quad (k = 0, 1, \ldots, n-1)$$

are roots of $P(x)$. Moreover, these numbers are distinct: if w and w' satisfy

$$w^n = w'^n = 1 \quad \text{and} \quad w - \frac{1}{w} = w' - \frac{1}{w'},$$

then

$$(w' - w)(1 + ww') = 0$$

and so $w = w'$ (note that $ww' \neq -1$ since n is odd). In addition, it is easy to see (from (1)) that $P(x)$ must be monic. Thus,

$$P(x) = \prod_{k=0}^{n-1} \left(x - 2i \sin \frac{2k\pi}{n} \right)$$

is the only possible solution.

Conversely, consider this polynomial $P(x)$ and let $n = 2m + 1$. Since

$$\sin \frac{2(m+k)\pi}{2m+1} = -\sin \frac{2(m-k+1)\pi}{2m+1} \quad \text{for} \quad k = 1, 2, \dots, m,$$

we have

$$P(x) = x \cdot \prod_{k=1}^{m} \left(x^2 + 4 \sin^2 \frac{2k\pi}{2m+1} \right)$$

and therefore $P(x)$ has real coefficients (and degree n). Now, for $x \neq 0$, we readily get

$$P\left(x - \frac{1}{x} \right) = \prod_{k=0}^{n-1} \frac{x^2 - 2ix \sin(2k\pi/n) - 1}{x}.$$

Since the roots of

$$x^2 - 2ix \sin(2k\pi/n) - 1$$

are

$$\exp\left(\frac{2k\pi i}{n} \right) \quad \text{and} \quad -\exp\left(\frac{-2k\pi i}{n} \right),$$

the polynomial $Q(x) = x^n P\left(x - \frac{1}{x}\right)$ is monic, of degree $2n$ and such that $Q(w) = 0$ when either $w^n = 1$ or $w^n = -1$. Since

$$w^{2n} - 1 = (w^n - 1)(w^n + 1),$$

the $2n$-th roots of unity are roots of $Q(x)$ and so $Q(x) = x^{2n} - 1$. It immediately follows that (1) holds.

Proving that a polynomial is the zero polynomial

Another consequence of our introductory result is, roughly speaking: a nonzero polynomial whose degree is not greater than n cannot have more than n roots. We give several examples, showing this result at work in various areas. Let us start with a simple exercise:

> Let n be an integer with $n \geq 2$ and $P_0(x), P_1(x), \dots, P_{n-2}(x)$ be polynomials of $\mathbb{C}[x]$. If $x^{n-1} + x^{n-2} + \cdots + x + 1$ divides the polynomial $P_0(x^n) + xP_1(x^n) + \cdots + x^{n-2}P_{n-2}(x^n)$, show that $x - 1$ divides each of the polynomials $P_0(x), P_1(x), \dots, P_{n-2}(x)$.

We observe that the roots of $x^{n-1}+x^{n-2}+\cdots+x+1$ are the numbers

$$w, w^2, \ldots, w^{n-1}, \quad \text{where} \quad w=\exp\left(\frac{2\pi i}{n}\right).$$

From the hypothesis, the polynomial

$$P(x)=P_0(x^n)+xP_1(x^n)+\cdots+x^{n-2}P_{n-2}(x^n)$$

satisfies

$$P(w^k)=0 \quad \text{for} \quad k=1,2,\ldots,n-1.$$

Since $(w^k)^n=1$, we deduce that

$$P_0(1)+w^kP_1(1)+w^{2k}P_2(1)+\cdots+w^{(n-2)k}P_{n-2}(1)=0, \quad k=1,2,\ldots,n-1$$

so that the $n-1$ complex numbers $w, w^2, \ldots, w^{n-1}$ are distinct roots of the polynomial

$$P_0(1)+xP_1(1)+x^2P_2(1)+\cdots+x^{n-2}P_{n-2}(1).$$

Thus, this polynomial is the zero polynomial, meaning that

$$P_0(1)=P_1(1)=\cdots=P_{n-2}(1)=0,$$

and these equalities imply the desired conclusion.

We continue with an example from linear algebra:

> Let A be an $n\times n$ matrix with complex entries and distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that $I_n, A, A^2, \ldots, A^{n-1}$ are independent vectors of $\mathcal{M}_n(\mathbb{C})$.

The hypothesis on A implies that A is similar to the diagonal matrix

$$D=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n),$$

that is, $A=PDP^{-1}$ for some invertible $n\times n$ matrix P.

Now, suppose that a relation

$$\alpha_0I_n+\alpha_1A+\cdots+\alpha_{n-1}A^{n-1}=O_n$$

holds for some complex numbers $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}$. Since $A^m=PD^mP^{-1}$ for all nonnegative integers m, we deduce that

$$\alpha_0I_n+\alpha_1D+\cdots+\alpha_{n-1}D^{n-1}=O_n.$$

As a result,

$$\alpha_0+\alpha_1\lambda_k+\alpha_2\lambda_k^2+\cdots+\alpha_{n-1}\lambda_k^{n-1}=0, \quad k=1,2,\ldots,n$$

and consequently $\lambda_1,\lambda_2,\ldots,\lambda_n$ are distinct roots of the polynomial

$$\alpha_0+\alpha_1x+\alpha_2x^2+\cdots+\alpha_{n-1}x^{n-1}.$$

This demands

$$\alpha_0=\alpha_1=\alpha_2=\cdots=\alpha_{n-1}=0$$

and the result follows.

Our next example is adapted from a problem of the 1996 Swedish competition:

> For every positive integer n, let $p_n(x) = \sum\limits_{k=0}^{n} \binom{2n}{2k} x^{2n-2k}(x^2-1)^k$. If m is a positive integer, prove that $p_m(p_n(x)) = p_{2mn}(x)$.

The key remark is the following one: for any real number $x \geq 1$, we have

$$p_n(x) = \frac{1}{2}\left((x+\sqrt{x^2-1})^{2n} + (x-\sqrt{x^2-1})^{2n}\right),$$

a direct consequence of the Binomial Theorem. Such an expression prompts us to examine what occurs when we substitute $\cosh t$ for x. We obtain that

$$\begin{aligned} p_n(\cosh t) &= \frac{1}{2}\left((\cosh t + \sinh t)^{2n} + (\cosh t - \sinh t)^{2n}\right) \\ &= \frac{1}{2}(e^{2nt} + e^{-2nt}) = \cosh(2nt) \end{aligned}$$

for all $t \in [0, \infty)$ and we may write

$$p_m(p_n(\cosh t)) = p_m(\cosh(2nt)) = \cosh(2m(2nt)) = \cosh(4mnt) = p_{2mn}(\cosh t).$$

Thus, the polynomial $p_m(p_n(x)) - p_{2mn}(x)$ takes the value 0 for infinitely many values of x and, as such, must be the zero polynomial. The desired relation follows.

Our last example is inspired by a problem that appeared in 2004 in *Mathematics Magazine* (problem 1688).

> Let p be an odd prime and let $Q(x)$ be a polynomial of degree $p-1$ with coefficients in $\mathbb{Z}_p$. Show that the mapping $x \mapsto Q(x)$ is not a bijection from $\mathbb{Z}_p$ onto itself.

We propose a solution that makes use of a Lagrange interpolation polynomial. Let us first refresh our memory about this polynomial. Let n be a nonnegative integer and let

$$x_1, x_2, \ldots, x_n, x_{n+1}, y_1, y_2, \ldots, y_n, y_{n+1}$$

be elements of the field F, the x_ks being distinct. Then, there exists a unique element of $F[x]$ of degree less than or equal to n taking the value y_k at x_k for $k = 1, 2, \ldots, n, n+1$.

Indeed, if we define $L_k(x) = \prod\limits_{j=1, j\neq k}^{n+1} (x - x_j)$, then the polynomial

$$L(x) = \sum_{k=1}^{n+1} y_k \cdot \frac{L_k(x)}{L_k(x_k)}$$

is clearly a solution (note that $L_j(x_k) = 0$ if $j \neq k$). In addition, if $P(x)$ is another solution, then $L(x) - P(x)$ is not of degree greater than n and $L(x_k) - P(x_k) = 0$ for $k = 1, 2, \ldots, n+1$; since the x_ks are distinct, we must have $L(x) - P(x) = 0$. Thus, $L(x)$ is the unique solution.

Returning to the problem, let

$$Q(x) = a_0 + a_1 x + \cdots + a_{p-1} x^{p-1},$$

where $a_0, a_1, \ldots, a_{p-1}$ are elements of $\mathbb{Z}_p = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{p-1}\}$ with $a_{p-1} \neq \overline{0}$ and assume that

$$(Q(\overline{0}), Q(\overline{1}), \ldots, Q(\overline{p-1}))$$

is a permutation of $(\overline{0}, \overline{1}, \ldots, \overline{p-1})$.

The polynomial $Q(x)$ is equal to the Lagrange interpolation polynomial $L(x)$ taking the values $Q(\overline{0}), Q(\overline{1}), \ldots, Q(\overline{p-1})$ at $\overline{0}, \overline{1}, \ldots, \overline{p-1}$, respectively, that is,

$$Q(x) = L(x) = \sum_{k=0}^{p-1} Q(\overline{k}) \cdot \frac{L_k(x)}{L_k(\overline{k})}, \text{ where } L_k(x) = \prod_{j=0, j\neq k}^{p-1} (x - \overline{j}),\ k = 0, 1, \ldots, p-1.$$

Since from Fermat's Little Theorem,

$$x^p - x = x(x - \overline{1})(x - \overline{2}) \cdots (x - \overline{p-1}),$$

the polynomial $\sum\limits_{k=0}^{p-1} L_k(x)$ is the derivative of $x^p - x$, which is the constant polynomial $-\overline{1}$; hence, $L_k(\overline{k}) = -\overline{1}$ (recall that $L_j(\overline{k}) = \overline{0}$ if $j \neq k$). As a result,

$$Q(x) = -\sum_{k=0}^{p-1} Q(\overline{k}) L_k(x),$$

and comparing the coefficients of x^{p-1},

$$a_{p-1} = -(Q(\overline{0}) + Q(\overline{1}) + \cdots + Q(\overline{p-1})).$$

Since

$$\{Q(\overline{0}), Q(\overline{1}), \ldots, Q(\overline{p-1})\} = \{\overline{0}, \overline{1}, \ldots, \overline{p-1}\}$$

by assumption, we arrive at $a_{p-1} = -(\overline{0} + \overline{1} + \cdots + \overline{p-1})$ and finally at the contradiction $a_{p-1} = \overline{0}$ (since the sum of the roots of $x^p - x$ is $\overline{0}$).

We conclude with two exercises.

Exercises

1. Let $p(x) \in \mathbb{R}[x]$ with $\deg(p(x)) \geq 2$. Prove that the graph of the function p cannot have more than one centre of symmetry.

2. Let n be a positive integer and let $a(x) \in \mathbb{R}[x]$ with $\deg(a(x)) = n$. Find $a(n+1)$ given that $a(k) = \frac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$,

a) using the polynomial $(x+1)a(x) - x$,

b) using a Lagrange interpolation polynomial.

The pqr Method: Part I

Steven Chow, Howard Halim and Victor Rong

Introduction

Each year, students who perform well on the International Mathematics Tournament of Towns are invited to the Tournament of Towns Summer Conference, to spend 10 days in Russia investigating current streams of mathematical research [1]. The three authors were participants of the 2016 Summer Conference, where they learned about the pqr method and its applications to solving inequalities.

The pqr method is a way to solve symmetric 3-variable inequalities by using three lemmas collectively known as the pqr lemmas. Given an inequality in terms of variables a, b, and c, we can make the substitutions $p = a + b + c$, $q = ab + bc + ca$, and $r = abc$. When two of p, q, r are fixed, the third obtains its maximum and minimum values when two of a, b, c are equal or one of a, b, c is 0. This reduces the problem to a 2-variable inequality.

This method is also known as the uvw method (although with slightly different variables) which was popularized by Knudsen [2].

Proving the pqr lemmas

The proof of the pqr lemma relies on a few lemmas whose proofs are left as an exercise for the reader. For three complex numbers a, b, c, let $p = a + b + c$, $q = ab + bc + ca$, $r = abc$, and define

$$T(p, q, r) = -4p^3r + p^2q^2 + 18pqr - 4q^3 - 27r^2 = (a - b)^2(b - c)^2(c - a)^2.$$

Lemma 1. If p, q, r are real numbers, then a, b, c are real numbers if and only if $T(p, q, r) \geq 0$.

Lemma 2. a, b, c are non-negative real numbers if and only if $p, q, r \geq 0$ and $T(p, q, r) \geq 0$.

The pqr Lemma. When we fix two of p, q, r such that there exists triples (p, q, r) satisfying $p, q, r \geq 0$ and $T(p, q, r) \geq 0$, the unfixed variable obtains its maximum and minimum values when two of a, b, c are equal. There is one exception – when r is the unfixed variable, its minimum value occurs when either two of a, b, c are equal, or one of them is equal to 0.

We will prove the r-lemma, the case when r is the unfixed variable. The p-lemma and q-lemma have similar proofs.

Proof. Fix $p = p_0$ and $q = q_0$. Then, $T(p_0, q_0, r)$ is a quadratic in r. Since its leading coefficient is negative, it points downwards, and the inequality $T(p_0, q_0, r) \geq 0$ defines a closed interval (the interval between both of its roots). The maximum value of r occurs at the right endpoint of the interval. Since $T(p_0, q_0, r) = 0$ at the endpoints, it follows that $(a-b)^2(b-c)^2(c-a)^2 = 0$, so two of a, b, c are equal.

Since we have a second inequality, $r \geq 0$, the minimum value of r occurs either at the left endpoint, or at $r = 0$ (if the left endpoint is negative). At the left endpoint, $T(p_0, q_0, r) = 0$, so two of a, b, c are equal. When $r = 0$, one of a, b, c equals 0. This completes the proof. □

The ability to use the *pqr* lemmas hinges on the property that any symmetric polynomial in terms of a, b, c can be written in terms of p, q, r. We will prove this property by demonstrating an algorithm applicable to any symmetric polynomial.

Proof. Call a symmetric polynomial *expressible* if it can be written in terms of p, q, r. Let $s_k = a^k + b^k + c^k$. First, we prove by induction that s_k is expressible for all non-negative integers k.

For the base case, we see that s_0, s_1, and s_2 are expressible, because $s_0 = 3$, $s_1 = p$, and $s_2 = p^2 - 2q$. Since $s_k = ps_{k-1} - qs_{k-2} + rs_{k-3}$, s_k is expressible for all non-negative integers k.

We see that

$$\begin{aligned} s_k s_l - s_{k+l} &= (a^k + b^k + c^k)(a^l + b^l + c^l) - (a^{k+l} + b^{k+l} + c^{k+l}) \\ &= a^k b^l + b^k c^l + c^k a^l + a^k c^l + b^k a^l + c^k b^l \\ &= \sum_{sym} a^k b^l. \end{aligned}$$

Therefore, $\displaystyle\sum_{sym} a^k b^l$ is expressible for all non-negative integers k, l. Finally, note that

$$\sum_{sym} a^k b^l c^m = r^n \sum_{sym} a^{k-n} b^{l-n} c^{m-n},$$

where k, l, m are positive integers and $n = \min(k, l, m)$. The sum on the right hand side is expressible, so the sum on the left is also expressible. It follows that all symmetric polynomials are expressible. □

Examples of the *pqr* Method

Example 1. Let a, b, c be non-negative real numbers. Prove that

$$a^5 + b^5 + c^5 + abc(ab + bc + ca) \geq a^2b^2(a+b) + b^2c^2(b+c) + c^2a^2(c+a).$$

Solution. Fix p and q. Let

$$f(r) = (7p^2 - 3q)r + p^5 - 5p^3q + 4pq^2,$$

then the inequality is equivalent to $f(r) \geq 0$. Since f is linear in terms of r, if the inequality holds for the extreme values of r, it holds for all values of r. From the r-lemma, the maximum and minimum of r occur when two of a, b, c are equal, or one of them is 0. WLOG, we can assume that either $a = b$ or $a = 0$.

When $a = 0$, the inequality is equivalent to $b^5 + c^5 \geq b^3c^2 + b^2c^3$. This inequality can be proved by summing up the two inequalities

$$\frac{b^5 + b^5 + b^5 + c^5 + c^5}{5} \geq b^3c^2 \quad \text{and} \quad \frac{b^5 + b^5 + c^5 + c^5 + c^5}{5} \geq b^2c^3,$$

both of which are true by the AM-GM inequality. When $a = b$, the inequality is equivalent to $c^5 + a^4c \geq 2a^2c^3$ after expanding. This inequality is true by the AM-GM inequality. Since we have proved the inequality when $a = 0$ and $a = b$, then we are done by the r-lemma.

Note that this inequality can also be proved by Schur's Inequality:

$$\sum_{cyc} a(a^2 - b^2)(a^2 - c^2) \geq 0.$$

However, the solution using the *pqr* lemma does not require any creative observations and is much easier to come up with.

Example 2. Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{9 - ab} + \frac{1}{9 - bc} + \frac{1}{9 - ca} \leq \frac{3}{8}.$$

Solution. Note that a, b, $c \leq 3$, so ab, bc, $ca \leq 9$, and all denominators are positive. Multiplying by $8(9 - ab)(9 - bc)(9 - ca)$ on both sides of the inequality, we obtain

$$\sum_{cyc} 8(9 - ab)(9 - bc) \leq 3(9 - ab)(9 - bc)(9 - ca).$$

Expanding yields

$$243 - 99\sum_{cyc} ab + 19\sum_{cyc} a^2bc - 3a^2b^2c^2 \geq 0$$

which is equivalent to

$$-3r^2 + 19pr - 99q + 243 \geq 0.$$

Note that p is already fixed by the constraint $a + b + c = 3$. Fix r as well. Let $f(q) = -3r^2 + 19pr - 99q + 243$. Since this is linear in terms of q, we only need to check the minimum and maximum values of q. By the q-lemma, these occur when $a = b$.

So it is enough to prove the inequality when $b = a$, $c = 3 - 2a$, and $0 \leq a \leq \frac{3}{2}$. Substituting these values and expanding yields

$$-4a^6 + 12a^5 - 9a^4 - 38a^3 + 156a^2 - 198a + 81 \geq 0.$$

This is equivalent to

$$-(a-1)^2(2a^2 - 3a + 9)(2a^2 + a - 9) \geq 0$$

which is true since

$$2a^2 - 3a + 9 = a^2 + (a - \frac{3}{2})^2 + \frac{27}{4}$$

and

$$2a^2 + a - 9 \leq 2(\frac{3}{2})^2 + \frac{3}{2} - 9 = -3 < 0$$

for $0 \leq a \leq \frac{3}{2}$.

Example 3. Let a, b, c be non-negative real numbers such that $a+b+c = 4$ and $a^2 + b^2 + c^2 = 6$. Prove that

$$a^6 + b^6 + c^6 \leq a^5 + b^5 + c^5 + 32.$$

Solution. The conditions are equivalent to $p = 4$ and $p^2 - 2q = 6$. With a little work, the inequality is equivalent to

$$p^6 - 6p^4q + 6p^3r + 9p^2q^2 - 12pqr - 2q^3 + 3r^2 \leq p^5 - 5p^3q + 5pq^2 + 5p^2r - 5qr + 32.$$

Note that from the conditions, p and q are already fixed with $p = 4$ and $q = 5$. Let

$$f(r) = -3r^2 + (12pq - 6p^3 + 5p^2 - 5q)r + p^5 - 5p^3q + 5pq^2 + 32 - p^6 + 6p^4q - 9p^2q^2 + 2q^3.$$

Since $f(r)$ is concave in terms of r, it is enough to prove the inequality for the minimum and maximum possible values of r. By the r-lemma, it is sufficient to consider the cases $a = 0$ and $a = b$.

If $a = 0$, there are no real solutions for b and c satisfying the conditions $a+b+c = 4$ and $a^2 + b^2 + c^2 = 6$. If $a = b$, solving the system of equations yields $a = b = 1$ and $c = 2$ or $a = b = \frac{5}{3}$ and $c = \frac{2}{3}$. These values satisfy the inequality, so the inequality is true.

It is important to note that one must be careful of the conditions when applying the pqr lemma. In the previous example, it should be noted that there are only two possible values of c when $a = b$ which obey the given conditions. The reason for this is because p and q were already fixed by the conditions, while usually p and q are arbitrarily fixed. However, this problem is not always so easily resolved. For example, consider the following problem:

> Let a, b, c be non-negative real numbers such that
>
> $$abc[(a-b)(b-c)(c-a)]^2 = 1.$$
>
> Find the minimum of $a + b + c$.

Even though the function $a+b+c$ is monotonic in terms of p, one cannot set $a=b$ or $a=0$ as that leads to a contradiction in the condition. If a problem has a similar issue when setting $a=b$ or $a=0$, then the pqr method cannot be used.

Problems

Here are a few problems which can be solved using the pqr method in a standard fashion.

Problem 1. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq\frac{2}{1+abc}.$$

Problem 2. Let a, b, c be non-negative real numbers such that $ab+bc+ca\neq 0$. Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\geq\frac{10}{(a+b+c)^2}.$$

Problem 3 ([3]). Let a, b, c be positive real numbers. Prove that

$$\sum_{cyc}\frac{ab(a+b)^2}{(c+a)(c+b)}\geq\frac{(a+b+c)^2}{3}.$$

Problem 4. Let a, b, c be real numbers such that a, b, $c\geq 1$ and $a+b+c=9$. Prove that

$$\sqrt{ab+bc+ca}\leq\sqrt{a}+\sqrt{b}+\sqrt{c}.$$

. . . .

To be continued.

References

[1] Doledenok, A., Fadin, M., Menshchikov, A., Semchankau, A. (2016, August). The pqr-method. Retrieved from `http://www.turgor.ru/lktg/2016/3/index.htm`

[2] Knudsen, M.B.T. (2009, May 26) The UVW-method [Online forum comment]. Message posted to `http://www.artofproblemsolving.com/community/c13188h278791`

[3] lebathanh (2016, October 4) inequalities [Online forum comment]. Message posted to `http://artofproblemsolving.com/community/c6h1314627`

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **January 1, 2018**.

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4241. *Proposed by Margarita Maksakova.*

Place the numbers $1, 2, \ldots, 11$ and some real number r on the edges of a cube so that at every vertex the sum of the numbers on the incident edges is the same. What is the smallest value of r for which this is possible?

4242. *Proposed by Mihály Bencze.*

Let $x_1 = 4$ and $x_{n+1} = [\sqrt[3]{2}x_n]$ for all $n \geq 1$, where $[\cdot]$ denotes the integer part function. Determine the largest positive $n \in \mathbb{N}$ for which x_n, x_{n+1}, x_{n+2} form an arithmetic progression.

4243. *Proposed by Dan Stefan Marinescu, Leonard Giugiuc and Hung Nguyen Viet.*

Let ABC be a triangle. Let I, r and R be the incenter, the inradius and the circumradius of ABC, respectively. Let D be the point of intersection of the line AI and the circumcircle of ABC. Similarly, define points E and F. Prove that

$$AD \cdot BE \cdot CF \geq 16rR^2.$$

4244. *Proposed by Michel Bataille.*

Let ABC be a triangle with no right angle and let H be its orthocenter. The parallel to BC through H intersects AB and AC at E and F and the perpendiculars through A to AB and AC at U and V, respectively. Let X and Y be the orthogonal projections of A onto BU and CV, respectively. Prove that E, F, X, Y are concyclic.

4245. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with circumcircle Γ, and let t be its tangent at A. Define T and E to be the points where the circle with centre B and radius BA again intersects t and AC, respectively, while T' and F are the points where the circle with centre C and radius CA intersects t and AB, respectively. If X is the point where TE and $T'F$ intersect, and Y is the second point where the line AX intersects Γ, prove that BC is the perpendicular bisector of the line segment XY.

4246. *Proposed by Leonard Giugiuc.*

Find the best lower bound for $abc + abd + acd + bcd$ over all positive a, b, c and d satisfying

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

4247. *Proposed by Missouri State University Problem Solving Group.*

Let B and C be two fixed points on a circle centered at O that are not diametrically opposite. Let A be a variable point on the circle distinct from B and C and not belonging to the perpendicular bisector of BC. Let M and N be the midpoints of the segments BC and AO, respectively. The line AM intersects the circle again at D, and finally, NM and OD intersect at P. Determine the locus of points P as A moves around the circle.

4248. *Proposed by Michel Bataille.*

Let n be a positive integer and let $p(x) = 1 + p_1(x) + p_2(x) + \cdots + p_n(x)$ where the polynomials $p_k(x)$ are defined by $p_0(x) = 2,\ p_1(x) = x^2 + 2$ and the recursion

$$p_{k+1}(x) = (x^2 + 2)p_k(x) - p_{k-1}(x)$$

for $k \in \mathbb{N}$. Find all the complex roots of $p(x)$.

4249. *Proposed by Daniel Sitaru.*

Let a, b, c be real numbers with at most one of them equal to zero. Prove that

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2 + b^2c^2 + c^2a^2} \leq 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

4250. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Let ABC be an acute angle triangle such that $\sin A = \sin B \sin C$. Prove that

$$\tan A \tan B \tan C \geq \frac{16}{3}.$$

. . . .

4241. *Proposé par Margarita Maksakova.*

À l'aide des nombres $1, 2, \ldots, 11$ et un certain nombre réel r, étiqueter les arêtes d'un cube de façon à ce que la somme des étiquettes des arêtes incidentes à un sommet soit la même, quel que soit le sommet. Quelle est la plus petite valeur possible pour r?

4242. *Proposé par Mihály Bencze.*

Soit $x_1 = 4$ et $x_{n+1} = [\sqrt[3]{2}x_n]$ pour $n \geq 1$, où $[\cdot]$ dénote la partie entière d'un nombre réel. Déterminer la plus grande valeur positive $n \in \mathbb{N}$ pour laquelle x_n, x_{n+1}, x_{n+2} forme une progression arithmétique.

4243. *Proposé par Dan Stefan Marinescu, Leonard Giugiuc et Hung Nguyen Viet.*

Soit ABC un triangle. Soient I, r et R le centre du cercle inscrit, le rayon du cercle inscrit et le rayon du cercle circonscrit de ABC, respectivement. Soit D le point d'intersection de la ligne AI et le cercle circonscrit de ABC. De façon similaire, définir les points E et F. Démontrer que

$$AD \cdot BE \cdot CF \geq 16rR^2.$$

4244. *Proposé par Michel Bataille.*

Soit ABC un triangle sans angle rectangle et soit H son orthocentre. La parallèle à BC passant par H intersecte AB et AC en E et F, et les perpendiculaires à AB et AC au point A en U et V, respectivement. Soient X et Y les projections orthogonales de A vers BU et CV, respectivement. Démontrer que E, F, X et Y sont cocycliques.

4245. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle aigu avec cercle circonscrit Γ et soit t sa tangente à A. Définissons T et E comme étant les points où le cercle avec centre B et rayon BA intersecte de nouveau t et AC, respectivement, puis T' et F comme étant les points où le cercle avec centre C et rayon CA intersecte de nouveau t et AB, respectivement. Si X est le point d'intersection de TE et $T'F$, et Y est le second point d'intersection de la ligne AX et Γ, démontrer que BC est la bissectrice orthogonale du segment XY.

4246. *Proposé par Leonard Giugiuc.*

Déterminer la meilleure borne inférieure pour $abc + abd + acd + bcd$ par rapport aux valeurs positives a, b, c et d satisfaisant

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

4247. *Proposé par Missouri State University Problem Solving Group.*

Soient B et C deux points sur un cercle centré à O, mais non diamétraux. Soit A un point variable sur le cercle, distinct de B et C et n'appartenant pas à la bissectrice orthogonale de BC. Soient M et N les mi points des segments BC et AO, respectivement. La ligne AM interscte de nouveau le cercle en D; enfin, NM

et OD intersectent en P. Déterminer le lieu géométrique des points P lorsque A se déplace sur le cercle.

4248. *Proposé par Michel Bataille.*

Soit n un entier positif et soit $p(x) = 1 + p_1(x) + p_2(x) + \cdots + p_n(x)$, où les polynômes $p_k(x)$ sont définis par $p_0(x) = 2,\ p_1(x) = x^2 + 2$ et la récursion

$$p_{k+1}(x) = (x^2 + 2)p_k(x) - p_{k-1}(x)$$

pour $k \in \mathbb{N}$. Déterminer les racines complexes de $p(x)$.

4249. *Proposé par Daniel Sitaru.*

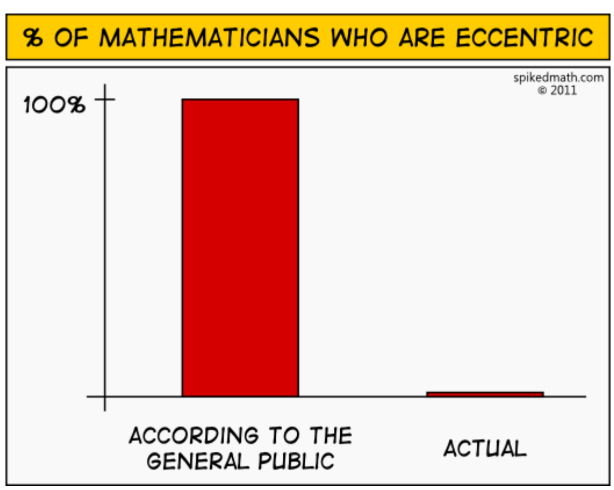
Soient a, b et c des nombres réels, dont au plus un seul est égal à zéro. Démontrer que

$$\frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2 + b^2c^2 + c^2a^2} \leq 2(a^2 + b^2 + c^2 - ab - bc - ca).$$

4250. *Proposé par Michael Rozenberg et Leonard Giugiuc.*

Soit ABC un triangle aigu tel que $\sin A = \sin B \sin C$. Démontrer que

$$\tan A \tan B \tan C \geq \frac{16}{3}.$$



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(5), p. 220–224.

4141. *Proposed by Leonard Giugiuc, Daniel Sitaru and Oai Thanh Dao; modified by the editor.*

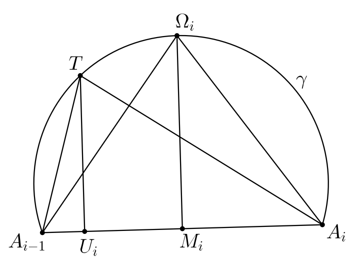
a) Let $A_0A_1 \dots A_{n-1}$ be a convex n-gon for which there exists an interior point T such that $\angle A_{i-1}TA_i = \frac{2\pi}{n}, i = 1, 2, \dots n$ (with $A_n \equiv A_0$). Construct regular n-gons Π_i externally on the sides $A_{i-1}A_i$. Prove that

$$[A_0A_1 \dots A_{n-1}] \leq \frac{1}{n}\sum_{i=1}^{n}[\Pi_i]$$

(where square brackets denote area).

b) $\star$ Does the inequality continue to hold if the given convex polygon is arbitrary?

Three solutions were received for part (a), and all were correct. Part (b) attracted no response, so it remains open. We present the solution to part (a) by Michel Bataille.



Let Ω_i be the reflection in $A_{i-1}A_i$ of the centre O_i of the polygon Π_i (see figure above). Because $\angle A_{i-1}TA_i = \angle A_{i-1}\Omega_iA_i = \frac{2\pi}{n}$, both T and Ω_i belong to the circular arc γ with endpoints A_{i-1} and A_i. Furthermore,

$$\frac{[\Pi_i]}{n} = [A_{i-1}O_iA_i] = [A_{i-1}\Omega_iA_i].$$

Let U_i and M_i denote the projections of T and Ω_i onto the line $A_{i-1}A_i$, respectively. Since Ω_i and M_i are on a diameter of the circle containing γ, we have $TU_i \leq \Omega_iM_i$. We deduce that

$$[A_{i-1}TA_i] = \frac{1}{2}A_{i-1}A_i \times TU_i \leq \frac{1}{2}A_{i-1}A_i \times \Omega_iM_i = [A_{i-1}\Omega_iA_i]$$

and so

$$[A_0A_1\dots A_{n-1}] = \sum_{i=1}^{n}[A_{i-1}TA_i] \le \sum_{i=1}^{n}[A_{i-1}\Omega_i A_i] = \sum_{i=1}^{n}\frac{1}{n}[\Pi_i]$$

and the required inequality follows.

4142. *Proposed by Daniel Sitaru.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$\left(1+\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{\frac{(a+b+c)^2}{a^2+b^2+c^2}} \le \left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right).$$

We received 4 correct solutions. We present the solution by Arkady Alt.

Assuming, due to the homogeneity of the original inequality, that $a+b+c=1$ and denoting

$$p = ab+bc+ca,\ \ q = abc,$$

we obtain

$$a^2+b^2+c^2 = 1-2p,$$

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right) = \frac{(a+b)\,(b+c)\,(c+a)}{abc} = \frac{p-q}{q},$$

and

$$1+\frac{a^2+b^2+c^2}{ab+bc+ca} = 1+\frac{1-2p}{p} = \frac{1-p}{p}.$$

The original inequality thus becomes

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{p}{q}-1.$$

Since $0 < q \le \dfrac{p^2}{3}$, we have $\dfrac{p}{q} \ge \dfrac{3}{p}$, and it suffices to prove the inequality

$$\left(\frac{1-p}{p}\right)^{\frac{1}{1-2p}} \le \frac{3}{p}-1.$$

For $0 < p \le \frac{1}{3}$, this is successively equivalent to

$$\frac{1-p}{p} \le \left(\frac{3-p}{p}\right)^{1-2p},$$

$$\left(\frac{3-p}{p}\right)^{2p} \le \frac{3-p}{1-p},$$

$$\left(\frac{3}{p}-1\right)^{2} \le \left(\frac{\frac{3}{p}-1}{\frac{1}{p}-1}\right)^{\frac{1}{p}}.$$

Denoting $t = \frac{1}{p} \in [3, \infty)$, we obtain the following more convenient equivalent form of the latter inequality.

$$(3t-1)^2 \leq \left(\frac{3t-1}{t-1}\right)^t \iff t \ln\left(\frac{3t-1}{t-1}\right) \geq 2\ln(3t-1).$$

Let

$$h(t) = t\left[\ln(3t-1) - \ln(t-1)\right] - 2\ln(3t-1).$$

Then

$$\begin{aligned} h'(t) &= \ln(3t-1) - \ln(t-1) + t\left(\frac{3}{3t-1} - \frac{1}{t-1}\right) - \frac{6}{3t-1} \\ &= \ln(3t-1) - \ln(t-1) - \frac{1}{t-1} - \frac{5}{3t-1} \end{aligned}$$

and

$$h''(t) = \frac{3}{3t-1} - \frac{1}{t-1} + \frac{1}{(t-1)^2} + \frac{15}{(3t-1)^2} = \frac{2\left(9t^2 - 14t + 7\right)}{(3t-1)^2 (t-1)^2}.$$

Since $h''(t) > 0$ for $t \geq 3$, $h'(t)$ increases on $[3, \infty)$ and, therefore,

$$h'(t) \geq h'(3) = \ln 8 - \ln 2 - \frac{1}{2} - \frac{5}{8} = 2\ln 2 - \frac{9}{8} > 0.$$

Hence, $h(t)$ increases on $[3, \infty)$ and, therefore,

$$h(t) \geq h(3) = 3(\ln 8 - \ln 2) - 2\ln 8 = 0.$$

Thus, $t\ln\left(\frac{3t-1}{t-1}\right) \geq 2\ln(3t-1)$, as desired.

4143. *Proposed by Roy Barbara.*

For any real number $x \geq 1$, let $y = x^{1/2} + x^{-1/2}$.

a) Express x in terms of y by a radical formula and check that no rational fraction $F(t)$ can exist such that $x = F(y)$. (A rational fraction is an expression of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials with rational coefficients.)

b) Find a closed form formula $x = F(y)$ containing no radicals.

c) $\star$ Is there a *complex* fraction such that $x = F(y)$? (A complex fraction is a function of the form $f(z)/g(z)$, where $f(t)$ and $g(t)$ are polynomials with complex coefficients.)

We received four solutions, all correct, and feature that of Joseph DiMuro.

a) We have $y = \frac{x+1}{\sqrt{x}}$, which can be rewritten as $x - y\sqrt{x} + 1 = 0$. The quadratic formula then gives us $\sqrt{x} = \frac{y \pm \sqrt{y^2-4}}{2}$. Both of these possible expressions for $\sqrt{x}$

are positive, and their product is 1, so exactly one of them is greater than 1. We are given that $\sqrt{x} \geq 1$, so we choose the larger solution; that is, $\sqrt{x} = \frac{y+\sqrt{y^2-4}}{2}$, and therefore,

$$x = \frac{1}{2}y^2 + \frac{1}{2}y\sqrt{y^2-4} - 1.$$

If, to the contrary, there were a rational fraction $F(t)$ such that $x = F(y)$, then x would be a rational number whenever y is a rational number. But when $y = 4$, we have $x = 7 + 4\sqrt{3}$, which is irrational. We conclude that no such rational fraction $F(t)$ exists.

b) Letting $\sec\theta = \frac{y}{2}$, we have $\tan\theta = \frac{\sqrt{y^2-4}}{2}$. Consequently,

$$\tan\left(\sec^{-1}\left(\frac{y}{2}\right)\right) = \frac{\sqrt{y^2-4}}{2},$$

which allows us to write $\frac{\sqrt{y^2-4}}{2}$ without any radicals, specifically

$$x = \frac{1}{2}y^2 + y\tan\left(\sec^{-1}\left(\frac{y}{2}\right)\right) - 1.$$

c) Assume that there is a complex fraction $F(y)$ such that $F(y) = \frac{1}{2}y^2 + \frac{1}{2}y\sqrt{y^2-4} - 1$. We then have

$$\sqrt{y^2-4} = \frac{F(y) - \frac{1}{2}y^2 + 1}{\frac{1}{2}y},$$

so $\sqrt{y^2-4}$ can itself be expressed as a complex fraction. Let

$$\sqrt{y^2-4} = \frac{f(y)}{g(y)},$$

where $f(y)$ and $g(y)$ are polynomials with complex coefficients. We may assume that the fraction is in lowest terms, so that $f(y)$ and $g(y)$ have no zeros in common. We then have

$$y^2 - 4 = \frac{(f(y))^2}{(g(y))^2}.$$

But this is impossible; all zeros of $\dfrac{(f(y))^2}{(g(y))^2}$ have even multiplicity, but the two zeros of $y^2 - 4$ (namely, ± 2) each have multiplicity 1. So *there can be no complex fraction $F(y)$ such that $x = F(y)$.*

Editor's Comments. The other formulas "containing no radicals" that we received were

$$x = \exp\left(2\cosh^{-1}\frac{y}{2}\right) \qquad \text{and} \qquad x = \cot^2\left(\frac{1}{2}\arcsin\frac{2}{y}\right).$$

These display, of course, simply cosmetic differences: radicals are historically important special cases of exponential functions.

4144. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c=1$. Find the maximum value of the expression

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3.$$

We received 19 submissions, all of which are correct. We present two solutions with the first one being a composite of very similar solutions by several solvers.

Solution 1, by Arkady Alt, Michel Bataille, Steven Chow, and Daniel Dan (independently).

Let $q=ab+bc+ca$ and $r=abc$. Then

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=1-2q$$

and

$$a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc=1-3q+3r.$$

Hence,

$$\begin{aligned}&\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3\\&=a^3+b^3+c^3-\frac{3}{2}(a^2+b^2+c^2)+\frac{3}{4}(a+b+c)-\frac{3}{8}\\&=1-3q+3r-\frac{3}{2}(1-2q)+\frac{3}{8}=3r-\frac{1}{8}=3abc-\frac{1}{8}\\&\le 3\left(\frac{a+b+c}{3}\right)^3-\frac{1}{8}=\frac{1}{9}-\frac{1}{8}=-\frac{1}{72}.\end{aligned}$$

Therefore, the required maximum $-\frac{1}{72}$ is attained exactly when $a=b=c=\frac{1}{3}$.

Solution 2, by Titu Zvonaru.

We prove that the searched maximum is $-\frac{1}{72}$.

For convenience, let $d=a^2b+ab^2+b^2c+bc^2+c^2a+ca^2$. Then

$$\left(a-\frac{1}{2}\right)^3+\left(b-\frac{1}{2}\right)^3+\left(c-\frac{1}{2}\right)^3\le-\frac{1}{72}$$

is equivalent in succession to

$$\begin{aligned}&9((2a-1)^3+(2b-1)^3+(2c-1)^3)\le 1,\\&9((a-b-c)^3+(b-c-a)^3+(c-a-b)^3)\le-(a+b+c)^3,\\&-9(a^3+b^3+c^3)-27d+162abc\le-(a^3+b^3+c^3)-3d-6abc,\\&8(a^3+b^3+c^3)+24d\ge 168abc,\\&a^3+b^3+c^3+3d\ge 21abc,\end{aligned}$$

which is true since by the AM-GM inequality we have

$$a^3 + b^3 + c^3 + 3d \geq 3abc + 3(6abc) = 21abc.$$

4145. *Proposed by Leonard Giugiuc.*

Prove that the system

$$\begin{cases} A^3 + A^2B + AB^2 + ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \\ \\ B^3 + B^2A + BA^2 + BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{cases}$$

has no solutions in the set of 3×3 matrices over complex numbers.

We received 11 correct solutions. We present the solution by AN-anduud Problem Solving Group.

Assume by contradiction that there exist matrices A and B that are solutions to this system.

$$\begin{cases} A^3 + A^2B + AB^2 + ABA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, & (1) \\ \\ B^3 + B^2A + BA^2 + BAB = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. & (2) \end{cases}$$

Equation (1) gives

$$A(A+B)^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(A) \cdot \det(A+B)^2 = 1$$
$$\Rightarrow \det(A+B) \neq 0. \qquad (3)$$

From equations (1) and (2), we get

$$(A+B)^3 = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det(A+B)^3 = 0 \Rightarrow \det(A+B) = 0,$$

which contradicts (3).

4146. *Proposed by Mehmet Berke Işler.*

Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 2(ab + bc + ca)$ and $\sqrt{a} + \sqrt{b} + \sqrt{c} = 2$. Prove that at least one of the numbers a, b, c is equal to 1.

There were seventeen correct solutions. Nine of them took the approach of Solution 1, two of Solution 2, two of Solution 3, and two of Solution 4. Below we present all four types of solutions by various solvers.

Solution 1.

Let $(a, b, c) = (x^2, y^2, z^2)$ with $x, y, z \geq 0$. Then

$$\begin{aligned} 0 &= 2(x^2y^2 + y^2z^2 + z^2x^2) - (x^4 + y^4 + z^4) \\ &= (x + y + z)(x + y - z)(y + z - x)(z + x - y) \\ &= 16(1 - z)(1 - x)(1 - y), \end{aligned}$$

from which at least one of x, y, z equals 1.

Solution 2.

The first condition can be rewritten as

$$\begin{aligned} 0 &= a^2 - 2(b + c)a + (b - c)^2 \\ &= [a - (\sqrt{b} - \sqrt{c})^2][a - (\sqrt{b} + \sqrt{c})^2]. \end{aligned}$$

This implies that $\sqrt{a} = \pm(\sqrt{b} - \sqrt{c})$ or $\sqrt{a} = \sqrt{b} + \sqrt{c}$, each of which implies that two of $\sqrt{a}, \sqrt{b}, \sqrt{c}$ add to the third. Thus, in view of the second condition, we are led to one of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ equal to 1.

Solution 3.

Since $(b + c - a)^2 = 4bc$, then $|b + c - a| = 2\sqrt{bc}$, from which $(\sqrt{b} \pm \sqrt{c})^2 = (\sqrt{a})^2$. Thus, as in the previous solution, two of the square roots add to the third and we reach the desired conclusion.

Solution 4.

With $(a, b, c) = (x^2, y^2, z^2)$, let $s = x + y + z = 2$, $q = xy + yz + zx$, $p = xyz$. Then

$$\begin{aligned} 0 &= 2(x^2y^2 + y^2z^2 + z^2x^2) - (x^4 + y^4 + z^4) \\ &= 4(x^2y^2 + y^2z^2 + z^2x^2) - (x^2 + y^2 + z^2)^2 \\ &= 4(q^2 - 2ps) - (s^2 - 2q)^2 = 4s^2q - 8ps - s^4 \\ &= 16(q - p - 1), \end{aligned}$$

whence $q = 1 + p$. Since x, y, z satisfy the equation

$$0 = t^3 - st^2 + qt - p = t^3 - 2t^2 + t + p(t - 1) = (t - 1)(t^2 - t + p),$$

one of x, y, z equals 1.

Editor's Comments. Note that, in the notation of Solutions 1 and 4, when $z = 1$, the second condition becomes $x + y = 1$ and the first condition is automatically satisfied. If x, y, z are sides of a triangle, the conditions state that the triangle has perimeter 2 and area 0, so that it is degenerate and the length of the longest side is equal to the sum of the lengths of the other sides.

4147. *Proposed by Mehtaab Sawhney.*

Let $\{a_i\}$ be a sequence of real numbers. Suppose that $|a_i - a_j| \geq 2^{i-j}$ if $i > j$, then find the minimal value of

$$\sum_{1\leq i<j\leq n} (a_j - a_i)^2.$$

The problem was solved by Steven Chow and the proposer. The following solution uses elements of both their solutions. An additional submission was incorrect.

We begin by showing that it suffices to consider only increasing sequences whose first term is 0. Let $\{a_1, \ldots, a_n\}$ be a sequence satisfying the conditions and let $\{b_1, \ldots, b_n\}$ be a reordering of its terms so that $b_1 < b_2 < \cdots < b_n$. Then $\sum(a_j - a_i)^2 = \sum(b_j - b_i)^2$ and $b_j - b_i \geq 2^{j-i}$ for $1 \leq i < j \leq n$.

The first conclusion is clear. As for the second, let $1 \leq i < j \leq n$. Consider the subsequence $B = \{b_i, b_{i+1}, \ldots, b_j\}$ with $j - i + 1$ entries. Let u be the minimum index and v be the maximum index for which a_u and a_v belong to B. Then $B \subseteq \{a_u, a_{u+1}, \ldots, a_v\}$, a set with $v - u + 1$ entries. Therefore $v - u \geq j - i$ and

$$b_j - b_i \geq |a_v - a_u| \geq 2^{|v-u|} \geq 2^{j-i}.$$

Since adding a constant to each term of a sequence does not alter either the given conditions or the square sum in the conclusion, we can, wolog, assume that $0 = a_1 < a_2 < \cdots < a_n$ so that $a_i \geq 2^{i-1}$ for $2 \leq i \leq n$.

Suppose now that $\{a_k\}$ is an increasing sequence with $a_1 = 0$ that minimizes the square sum. We show that $a_n = 2^{n-1}$.

Suppose, if possible, that $a_i > 2^{i-1}$ for $i \geq 2$. Let $m = \min\{a_i - 2^{i-1} : 2 \leq i \leq n\}$; note that $m > 0$. Suppose that $b_1 = a_1 = 0$ and $b_i = a_i - m$ for $2 \leq i \leq n$. Then $b_j < a_j$ and $b_j \geq a_j - (a_j - 2^{j-1}) = 2^{j-1}$ for $j \geq 2$; also $b_j - b_i = a_j - a_i$ for $j > i \geq 2$. Thus $\{b_i\}$ satisfies the conditions of the problem and the square sum is strictly smaller, contradicting the mimimality of $\{a_k\}$. Therefore, there is at least one value of k for which $a_k = 2^{k-1}$.

Let k be the largest index with $a_k = 2^{k-1}$. If $k < n$, suppose that $m = \min\{a_j - 2^{j-1} : k + 1 \leq i \leq n\}$. Define $b_i = a_i$ for $1 \leq i \leq k$ and $b_j = a_j - m$ for $k + 1 \leq j \leq m$. When $1 \leq i \leq k < j \leq n$,

$$\begin{aligned} b_j - b_i = b_j - b_k + (b_k - b_i) &\geq 2^{j-1} - 2^{k-1} + 2^{k-i} \\ &= 2^{j-i} + (2^{j-i} - 2^{k-i})(2^{i-1} - 1) \geq 2^{j-i}. \end{aligned}$$

It is now clear that $\{b_i\}$ satisfies the conditions of the problem and that the square sum is strictly smaller. This again contradicts minimality, and so $a_n = 2^{n-1}$.

The minimizing sequence exhibits a kind of bilateral symmetry. Let $\{a_k\}$ be such a sequence, with $x_k = a_{k+1} - a_k > 0$ for $1 \le k \le n-1$. Then for $1 \le i < j \le n$,

$$\begin{aligned}(a_j - a_i)^2 + (a_{n+1-i} - a_{n+1-j})^2 &\ge 2\left[\frac{(a_j - a_i) + (a_{n+1-i} - a_{n+1-j})}{2}\right]^2 \\ &= 2\left[\sum_{k=i}^{j-1} \frac{x_k + x_{n-k}}{2}\right]^2 = 2\left[\sum_{k=i}^{j-1} y_k\right]^2 \\ &= \left(\sum_{k=i}^{j-1} y_k\right)^2 + \left(\sum_{k=n-(j-1)}^{n-i} y_k\right)^2\end{aligned}$$

where $y_k = \frac{1}{2}(x_k + x_{n-k}) = y_{n-k}$ for $1 \le k \le n-1$.

Define $b_1 = a_1 = 0$, and $b_k = y_1 + y_2 + \cdots + y_{k-1}$ for $1 \le k \le n-1$. Then, for $1 \le i < j \le n$,

$$\begin{aligned}b_j - b_i = y_i + \cdots + y_{j-1} &= \frac{1}{2}\left[(x_i + \cdots + x_{j-1}) + (x_{n-j+1} + \cdots + x_{n-i})\right] \\ &= \frac{1}{2}\left[(a_j - a_i) + (a_{n+1-i} - a_{n-j+1})\right] \\ &\ge 2^{j-i},\end{aligned}$$

$b_n = a_n = 2^{n-1}$ and

$$\sum_{1 \le i < j \le n} (b_j - b_i)^2 \le \sum_{1 \le i < j \le n} (a_j - a_i)^2.$$

Since $\{a_k\}$ was minimal, we must have $a_k = b_k$ for each k.

We have a final step. We subtract from each term of our optimal sequence the number 2^{n-2} to get a *balanced* sequence $\{a_k : 1 \le k \le n\}$ which satisfies the conditions of the problem along with the condition that

$$a_k = -2^{n-2} + y_1 + y_2 + \cdots + y_{k-1} = -(2^{n-2} - y_1 - y_2 - \cdots - y_{k-1}) = -a_{n+1-k}$$

for $2 \le k \le n-1$; in particular $a_n = 2^{n-2} = -a_1$. Note in particular that $|2a_k| = |a_k - a_{n+1-k}| \ge 2^{n-2k}$ for $1 \le k \le n$.

It is now time to introduce the minimizing candidates. Let $k \ge 1$ and define the balanced sequences $A_n = \{a_1, \ldots, a_n\}$ by

$$A_{2m} = \{-4^{m-1}, -4^{m-2}, \ldots, -4^0 = -1, 4^0 = 1, \ldots, 4^{m-2}, 4^{m-1}\},$$

$$\begin{aligned}A_{2m+1} &= \{-2^{2m-1}, -2^{2m-3}, \ldots, -2, 0, 2, \ldots, 2^{2m-3}, 2^{2m-1}\} \\ &= 2\{-4^{m-1}, -4^{m-2}, \ldots, -1, 0, 1, \ldots, 4^{m-2}, 4^{m-1}\}.\end{aligned}$$

It is readily checked that each A_n has n terms and satisfies the conditions of the problem; in crucial circumstances it does so with equality, to wit

$$a_{m+k} - a_{m+1-k} = 2(4^{k-1}) = 2^{2k-1}$$

when $n = 2m$ and $1 \le k \le m$, and

$$a_{m+1+k} - a_{m-k} = 2(4^{k-1}) = 2^{2k-1}$$

when $n = 2m+1$ and $1 \le k \le m$. We will evaluate the square sum for these sequences.

Let S_m be the sum $\sum_{1\le i<j\le 2m}(a_j - a_i)^2$ for the sequence A_{2m}.

Noting that the sequence $A_{2(m+1)}$ consists of the sequence A_{2m} with additional terms $\pm 4^k$ appended at the ends, we see that, for $m \ge 1$,

$$\begin{aligned}
S_{m+1} &= S_m + (4^m + 4^m)^2 + 2\sum_{k=0}^{m-1}\left[(4^m - 4^k)^2 + (4^m + 4^k)^2\right] \\
&= S_m + 4^{2m+1} + m\cdot 4^{2m+1} + 4\left(\frac{4^{2m}-1}{15}\right) \\
&= S_m + m\cdot 4^{2m+1} + \frac{1}{15}\left(4^{2m+3} - 4\right).
\end{aligned}$$

Hence

$$\begin{aligned}
S_m &= S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots + (S_m - S_{m-1}) \\
&= 4 + 4^3[1 + 2\cdot 16 + 3\cdot 16^2 + \cdots + (m-1)16^{m-2}] \\
&\quad + \frac{4^5}{15}[1 + 16 + \cdots + 16^{m-2}] - \frac{4(m-1)}{15} \\
&= 4 + \frac{4^3}{15^2}[(m-1)16^m - m\cdot 16^{m-1} + 1 + (16^m - 16)] - \frac{4(m-1)}{15} \\
&= \frac{4m(16^m - 1)}{15}.
\end{aligned}$$

Thus, when $n = 2m$ is even, the sum $\sum(a_j - a_i)^2$ is equal to

$$\frac{4m(16^m - 1)}{15} = \frac{2n(4^n - 1)}{15}.$$

When $n = 2m+1$ is odd, then $\sum(a_j - a_i)^2$ is equal to

$$\begin{aligned}
4S_m + 4\cdot 2(1 + 16 + \cdots + 16^{m-1}) &= \frac{1}{15}[16m(16^m - 1) + 8(16^m - 1)] \\
&= \frac{8}{15}[(2m+1)(4^{2m} - 1)] \\
&= \frac{8n}{15}(4^{n-1} - 1).
\end{aligned}$$

We now establish minimality. Clearly, $\{-1, 1\}$ is an optimizing sequence with two entries. Suppose we have an optimizing sequence for $n = 2m > 2$, namely the balanced

$$A = \{-4^{m-1}, a_2, a_3, \ldots, a_{2m-1}, 4^{m-1}\}.$$

Denoting by S the sum of all the squares of differences that do not involve the first and last terms, we find that the sum of the squares of the differences equals

$$\begin{aligned}
&S + (2 \cdot 4^{m-1})^2 + 2 \sum_{k=2}^{2m-1} (4^{m-1} - a_k)^2 \\
&= S + 4^{2m-1} + 4(m-1)4^{2m-2} - \left(4^m \sum_{k=2}^{m} (a_k + a_{2m+1-k})\right) + 2 \sum_{k=2}^{2m-1} a_k^2 \\
&= T + 2 \sum_{k=-(m-2)}^{m-1} a_{m+k}^2 \\
&\geq T + 2 \sum_{k=2}^{m} (4^k),
\end{aligned}$$

where T is a constant, independent of the balanced sequence. In fact, by inserting $A_{2(k-1)}$ between -4^{k-1} and 4^{k-1}, we get equality at the last stage of the display. So we can work our way up from A_2 to find that A_{2m} is an optimizing sequence for each positive integer m.

We can follow a similar argument to show that A_{2n+1} is also optimizing for each positive integer m. Thus the minimum value of the square sum is

$$\frac{2n(4^n - 1)}{15}, \quad \text{when } n \text{ is even},$$
$$\frac{8n(4^{n-1} - 1)}{15}, \quad \text{when } n \text{ is odd}.$$

This can also be rendered as

$$\frac{2n}{15}\left(4^n + (-1)^n \frac{3}{2} - \frac{5}{2}\right)$$

for all n.

4148. *Proposed by Lorian Saceanu.*

For positive real numbers x, y and z, show that

$$\begin{aligned}
&\sqrt{xy(x+y)} + \sqrt{yz(y+z)} + \sqrt{xz(x+z)} \\
&\qquad \geq \sqrt{(x+y)(y+z)(z+x)} + (x+y+z)\sqrt{\frac{2xyz}{3(xy+yz+xz)}}.
\end{aligned}$$

We received four correct submissions, out of which we present the solution by Oliver Geupel.

We start with the following calculation:

$$\begin{aligned}
&\left(\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{zx(z+x)}\right)\cdot\left(\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\right)\\
&= x\sqrt{y(x+y)(y+z)}+y\sqrt{x(x+y)(x+z)}+\sqrt{xyz}(x+y)\\
&\quad+\sqrt{xyz}(y+z)+y\sqrt{z(y+z)(x+z)}+z\sqrt{y(y+z)(x+y)}\\
&\quad+x\sqrt{z(z+x)(y+z)}+\sqrt{xyz}(x+z)+z\sqrt{x(x+y)(x+z)}\\
&= \sqrt{y(x+y)(y+z)}(x+z)+\sqrt{x(x+y)(x+z)}(y+z)\\
&\quad+\sqrt{z(x+z)(y+z)}(x+y)+2\sqrt{xyz}(x+y+z)\\
&= \sqrt{(x+y)(y+z)(z+x)}\left(\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}\right)\\
&\quad+2(x+y+z)\sqrt{xyz}.
\end{aligned}$$

Rearranging, we get as a consequence that

$$\begin{aligned}
&\sqrt{xy(x+y)}+\sqrt{yz(y+z)}+\sqrt{zx(z+x)}\\
&= \sqrt{(x+y)(y+z)(z+x)}+(x+y+z)\frac{\sqrt{4xyz}}{\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)}}.
\end{aligned}$$

By Jensen's inequality for the square root function, we have

$$\begin{aligned}
\sqrt{x(y+z)}+\sqrt{y(z+x)}+\sqrt{z(x+y)} &\le 3\sqrt{\frac{x(y+z)+y(z+x)+z(x+y)}{3}}\\
&= \sqrt{6(xy+yz+zx)}.
\end{aligned}$$

The desired result immediately follows. Note that from the properties of Jensen's inequality, the equality holds if and only if

$$x(y+z)=y(z+x)=z(x+y),$$

that is, if and only if $x=y=z$.

4149. *Proposed by Daniel Sitaru.*

Prove that if $[a,b]\subset\left[0,\frac{\pi}{4}\right]$ then:

$$3(a\tan b+b\tan a)\ge ab(6+a\tan a+b\tan b).$$

We received five submissions, all correct. We present the solution by Digby Smith.

We first prove that if x is a real number such that $0\le x\le 1$, then

$$(3-x^2)\tan x\ge 3x. \tag{1}$$

From the Maclaurin series expansion for $\tan x$, we have that

$$\tan x \geq x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7.$$

Hence,

$$\begin{aligned}
&(3 - x^2)\tan x - 3x \\
&\geq (3 - x^2)\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7\right) - 3x \\
&= \left(3x + x^3 + \frac{6}{15}x^5 + \frac{51}{315}x^7\right) - \left(x^3 + \frac{1}{3}x^5 + \frac{2}{15}x^7 + \frac{17}{315}x^9\right) - 3x \\
&= \frac{1}{15}x^5 + \frac{9}{315}x^7 - \frac{17}{315}x^9 \\
&= \frac{1}{315}x^5(21 + 9x^2 - 17x^4) \geq 0,
\end{aligned}$$

which establishes (1).

Applying (1) with $x = a$ and b, respectively, we then have

$$(3 - a^2)\tan a \geq 3a \quad \text{and} \quad (3 - b^2)\tan b \geq 3b.$$

Therefore,

$$a(3 - b^2)\tan b + b(3 - a^2)\tan a \geq 6ab,$$

from which the given inequality follows immediately.

Editor's comment. Roy Barbara also proved (1) first and then used calculus with some elaborate calculations to actually show that the given inequality holds for all $a, b \in [0, \pi/2)$.

4150. *Proposed by Leonard Giugiuc.*

Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty}\left(x_n^2 + 2x_n + \frac{32}{x_n^3}\right) = 12.$$

Show that $\lim_{n\to\infty} x_n$ exists and find its value.

We received twelve correct and complete submissions. We present two solutions.

Solution 1, by the AN-anduud Problem Solving Group.

The statement in the question is equivalent to:

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \left|x_n^2 + 2x_n + \frac{32}{x_n^3} - 12\right| < \varepsilon.$$

We calculate

$$\begin{aligned}\varepsilon &> \left|x_n^2 + 2x_n + \frac{32}{x_n^3} - 12\right| \\ &= \left|\frac{(x_n-2)^2(x_n^3+6x_n^2+8x_n+8)}{x_n^3}\right| \\ &= (x_n-2)^2\left(1+\frac{6}{x_n}+\frac{8}{x_n^2}+\frac{8}{x_n^3}\right).\end{aligned}$$

Since the right factor is greater than 1, we conclude that for all $n \geq n_0$: $(x_n-2)^2 < \varepsilon$ or $|x_n - 2| < \sqrt{\varepsilon}$. Thus

$$\lim_{n\to\infty} x_n = 2.$$

Solution 2, by C.R. Pranesachar.

We will use the following observation: If $\langle a_n \rangle$ and $\langle b_n \rangle$ are two real sequences such that $\lim_{n\to\infty} a_n = 0$ and $\langle b_n \rangle$ is bounded then

$$\lim_{n\to\infty} a_n b_n = 0.$$

From the limit given in the question we conclude that $\langle x_n \rangle$ has to be a bounded sequence, and thus $\langle x_n^3 \rangle$ is bounded as well. Using our observation we then get

$$\begin{aligned}0 &= \lim_{n\to\infty}\left(x_n^2 + 2x_n + \frac{32}{x_n^3} - 12\right)x_n^3 \\ &= \lim_{n\to\infty}\left(x_n^5 + 2x_n^4 - 12x_n^3 + 32\right) \\ &= \lim_{n\to\infty}(x_n-2)^2\left(x_n^3+6x_n^2+8x_n+8\right).\end{aligned}$$

Since $x_n > 0$, we have

$$0 < \frac{1}{x_n^3+6x_n^2+8x_n+8} < \frac{1}{8},$$

and so again by the above observation

$$\begin{aligned}0 &= \lim_{n\to\infty}\left((x_n-2)^2\left(x_n^3+6x_n^2+8x_n+8\right)\right)\cdot\frac{1}{x_n^3+6x_n^2+8x_n+8} \\ &= \lim_{n\to\infty}(x_n-2)^2,\end{aligned}$$

which yields $\lim_{n\to\infty} x_n = 2$.

AUTHORS' INDEX

Solvers and proposers appearing in this issue
(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations