THE CONTEST CORNER

No. 55

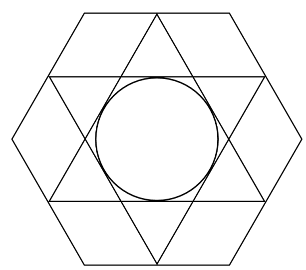
John McLoughlin

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **January 1, 2018**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

CC271. Warren's lampshade has an interesting design. Within a regular hexagon (six sides) are two intersecting equilateral triangles, and within them is a circle which just touches the sides of the triangles. (See the diagram.) The points of the triangles are at the midpoints of the sides of the hexagons.



If each side of the hexagon is 20 cm long, find:

a) the area of the hexagon;

b) the area of each large equilateral triangle;

c) the area of the circle.

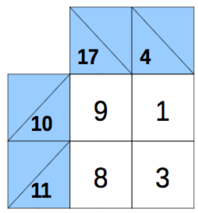
CC272. A *sum-palindrome number* (SPN) is a number that, when there are an even number of digits, the first half of the digits sums to the same total as the second half of the digits, and when odd, the digits to the left of the central digit sum to the same total as the digits to the right of the central digit. A *product-palindrome number* (PPN) is like a sum-palindrome, except the products of the digits are involved, not the sums.

a) How many three-digit SPNs are there?

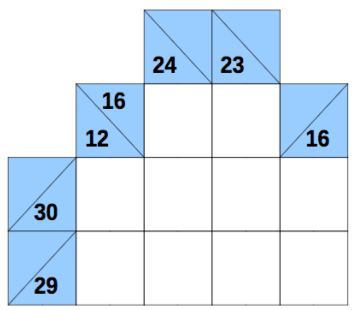
b) The two SPNs 1203 and 4022 sum to 5225, which is itself a SPN. Is it true that, for any two four-digit SPNs less than 5000, their sum is also a SPN?

c) How many four-digit non-zero PPNs are there?

CC273. Kakuro is the name of a number puzzle where you place numbers from 1 to 9 into empty boxes. There are three rules in a Kakuro puzzle: only numbers from 1 to 9 may be used, no number is allowed in any line (across or down) more than once, the numbers must add up to the totals shown at the top and the left. The diagram below shows a small finished Kakuro puzzle:



Solve the following Kakuro puzzle. Is your solution unique?



CC274. In his office, Shaquille had nine ping pong balls which he used for therapeutic recovery by throwing them into the waste basket at slack times. Each time he threw the nine balls, some of them would land in the basket, with the rest of them landing on the floor.

a) If the balls are identical, how many different results could there be?

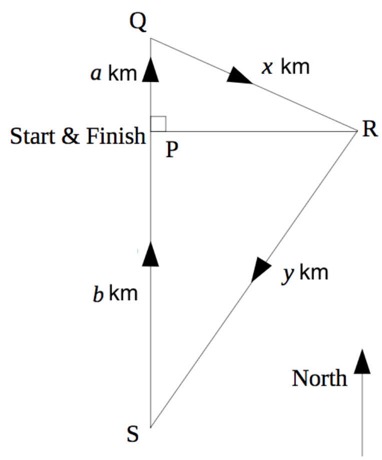
b) Suppose now that the balls are numbered 1 to 9. How many different results could there be now? (For example one possible result is for balls 1 to 4 to land in the basket, with 5 to 9 on the floor.)

c) Suppose instead that the balls are not numbered, but five are coloured yellow and four blue. Now how many different results could there be? (For example one possible result is for two yellow balls and three blue balls to land in the basket, and the rest to land on the floor.)

d) One day another basket appeared in the office. So now Shaquille had a choice of baskets to aim at. How did this change the answers to (a), (b), and (c)?

e) Now suppose that every time he threw the balls at the two baskets, each basket received at least two balls. How would this change the answers to (a), (b), and (c)?

CC275. The local sailing club is planning its annual race. By tradition the boats always start at P, sailing due North for a distance of a km until they reach Q. They then turn and sail a distance of x km to R (which is due East of P). Next they turn and sail a distance of y km to S (which is South of P) before finally sailing due North for a distance of b km until the finish line, which is back at the starting point P (see the diagram).



Bernie, the Club Commander, makes four extra rules for this year's race:

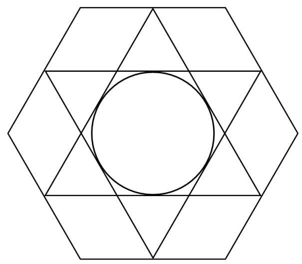
- $a + b = 40$,
- $x + y = 50$,
- $a < b$,
- the four lengths (a, b, x, y) must each be a whole number of kilometres. (Bernie doesn't like decimals.)

a) Find four numbers (a, b, x, y) which satisfy Bernie's four rules.

b) Are there four different numbers (a, b, x, y), which also satisfy Bernie's four rules apart from the four numbers you found in part (a)? Explain.

. . . .

CC271. Un abat-jour a un motif intéressant. À l'intérieur d'un hexagone régulier se trouvent deux triangles équilatéraux qui se coupent et à l'intérieur de ceux-ci, il y a un cercle qui touche aux côtés des triangles. (Voir la figure ce-dessous.) Les sommets des triangles sont les milieux des côtés de l'hexagone.



Sachant que chaque côté de l'hexagone a une longueur de 20 cm, déterminer:

a) l'aire de l'hexagone;

b) l'aire de chaque grand triangle équilatéral;

c) l'aire du cercle.

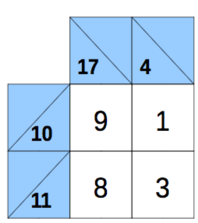
CC272. Un *nombre palindrome additif* (NPA) est un entier non négatif dont la somme de la première moitié de ses chiffres est égale à la somme de la seconde moitié de ses chiffres (s'il admet un nombre pair de chiffres) et s'il admet un nombre impair de chiffres, la somme des chiffres qui précèdent le chiffre central est égale à la somme des chiffres qui suivent le chiffre central. Un *nombre palindrome multiplicatif* (NPM) est défini de façon semblable, la somme étant remplacée par le produit.

a) Combien y a-t-il de NPA de trois chiffres?

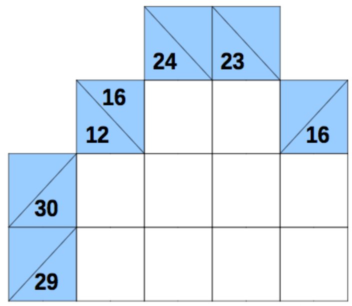
b) Le NPA 1203 et le NPA 4022 ont une somme de 5225, qui est un NPA. Est-il vrai qu'étant donné deux NPA de quatre chiffres, inférieurs à 5000, leur somme est toujours un NPA?

c) Combien y a-t-il de NPM de quatre chiffres?

CC273. Le Kakuro est un jeu logique comprenant une grille de cases dans lesquelles on place des chiffres de 1 à 9 pour former des nombres qui se lisent de gauche à droite ou de haut en bas. Le jeu comporte trois règles: seuls les chiffres de 1 à 9 peuvent être utilisés, un nombre ne peut contenir un même chiffre plus d'une fois, les chiffres d'un nombre doivent avoir une somme égale au nombre indiqué dans la case à la gauche ou au haut du nombre. La figure suivante montre un jeu Kakuro terminé:



Résoudre le jeu Kakuro suivant. La solution est-elle unique?



CC274. Dans son bureau, Shaquille s'amuse avec neuf balles de ping-pong qu'il tente de lancer dans une poubelle. Cela lui sert de thérapie par le jeu. Chaque fois qu'il lance les neuf balles, certaines d'entre elles tombent dans la poubelle, tandis que les autres se retrouvent par terre.

a) Sachant que les balles sont identiques, combien de résultats différents peut-il y avoir?

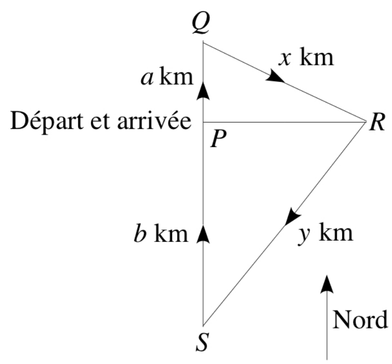
b) En supposant que les balles sont numérotées de 1 à 9, combien de résultats différents peut-il y avoir maintenant? (Par exemple, on peut considérer comme résultat possible que les balles de 1 à 4 tombent dans la poubelle et que les balles de 5 à 9 se retrouvent par terre.)

c) Supposons que les balles ne sont pas numérotées, mais que cinq d'entre elles sont jaunes et quatre sont bleues. Combien de résultats différents peut-il y avoir maintenant? (Par exemple, on peut considérer comme résultat possible que deux balles jaunes et trois balles bleues tombent dans la poubelle et que les autres se retrouvent par terre.)

d) Un bon jour, Shaquille hérite d'une deuxième poubelle pour son bureau. Il peut donc viser l'une ou l'autre poubelle. Comment cela change-t-il les réponses des parties (a), (b) et (c)?

e) Supposons qu'à chaque fois que Shaquille lance les neuf balles, chaque poubelle reçoit au moins deux balles. Comment cela change-t-il les réponses des parties (a), (b) et (c)?

CC275. Un club nautique prépare sa course d'hiver. Par tradition, les voiliers partent d'un point P et naviguent plein nord sur une distance de a km jusqu'au point Q. Ils changent de cap et parcourent x km en ligne droite jusqu'au point R situé à l'est de P. Ils changent de cap et parcourent y km en ligne droite jusqu'au point S situé au sud de P, pour ensuite naviguer plein nord sur une distance de b km jusqu'au point d'arrivée P. (Voir la figure ci-dessous.)



Bernard, le commandant du club nautique, ajoute quatre règlements:

- $a + b = 40$,
- $x + y = 50$,
- $a < b$,
- chacun des nombres a, b, x et y doit être un nombre entier strictement positif. (Bernard n'aime pas les nombres décimaux.)

a) Déterminer quatre nombres, a, b, x et y, qui vérifient les règlements de Bernard.

b) Y a-t-il quatre autres nombres, a, b, x et y, qui vérifient les règlements de Bernard autres que ceux déterminés dans la partie (a)? Expliquer.

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 196–198.

CC221. What is the smallest positive integer n such that if S is any set containing n or more integers, then there must be three integers in S whose sum is divisible by 3?

Originally Question 29 from 2001 High School Math Contest of University of South Carolina.

We received twelve solutions, all of which were correct.

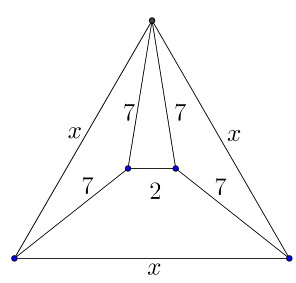
If there are three integers that have a sum divisible by three, then either

a) all three are congruent modulo 3, or

b) their residues modulo 3 must be 0, 1, and 2.

It is possible to avoid either of these scenarios in a set of three integers; for example, the elements in $\{3, 4, 6\}$ have residues (mod 3) of $\{0, 1, 0\}$ or in a set of four integers (take $\{3, 4, 6, 7\}$, with residues (mod 3) of $\{0, 1, 0, 1\}$). A set of five integers must, by the pigeonhole principle, either have three elements with the same residue (mod 3) (case (a)) or three elements with residues (mod 3) of 0, 1, and 2 (case (b)). Thus the smallest set is five.

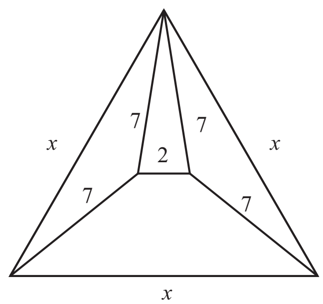
CC222. What is the value of x in the plane figure shown?

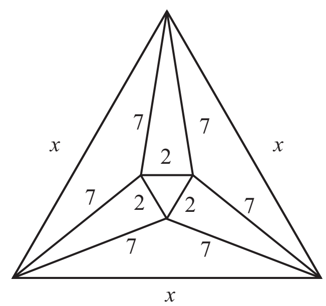


Originally Question 30 from 2002 High School Math Contest of University of South Carolina.

We received 14 correct and complete solutions, out of which we present the one by Ángel Plaza.

The figure below shows that by symmetry we can add other line segments and their known lengths.





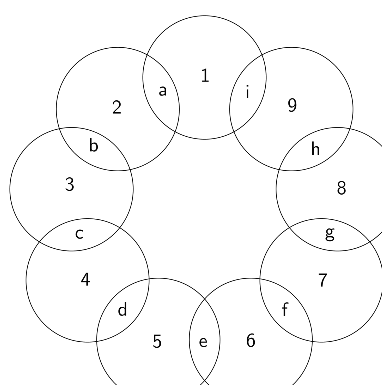
The area of the large equilateral triangle (in terms of x) may now be obtained in two ways, by directly calculating the area of the triangle, or by calculating the area of each of the sub-triangles. Thus

$$\frac{x^2\sqrt{3}}{2} = 3\left(\frac{x}{2}\sqrt{49 - \frac{x^2}{4}}\right) + 3(\sqrt{48}) + \sqrt{3}.$$

The only positive solution of this equation is $x = 13$.

Editor's comments. For an alternative solution inspired by the symmetry of the diagram on the right, note that in an equilateral triangle with sidelength x the distance from a corner to the center is $\frac{x}{\sqrt{3}}$, two thirds of the height. As the center of the large equilateral triangle is the center of the inner equilateral triangle as well, we can also evaluate this distance as $\sqrt{48} + \frac{1}{\sqrt{3}} = \frac{13}{\sqrt{3}}$, which immediately gives $x = 13$.

CC223. The letters a, b, c, d, e, f, g, h and i in the figure below represent the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in a certain order. In each of the nine circles, we sum the three numbers so that nine sums are obtained. Suppose that all nine sums are equal. What is the value of $a + d + g$?



Originally Question 28 from 2003 High School Math Contest of University of South Carolina.

We received nine solutions, all of which were correct. We present an editor's amalgamation of the versions submitted by Somasundaram Muralidharan and Ricard Peiró i Estruch.

The sum of the numbers in the nine circles is

$$(1+2+\cdots+9)+2\cdot(a+b+\cdots+i)=3\cdot 45=135.$$

Since each circle contains the same sum, the sum in each circle must be $135/9=15$.

The circle holding the number 1 gives us $1+a+i=15$, so $a+i=14$. Since every letter must take on an integer value between 1 and 9 inclusive, the only possible pairs for (a,i) are $\{(6,8),(8,6),(5,9),(9,5)\}$. The circle to the right gives us $i+9+h=15$, so $i+h=6$, which implies $i\le 5$. The only possibility for the pair (a,i) is $(9,5)$. Then $h=1$.

We can now continue around the circuit clockwise, filling in one new letter as we consider each new circle. The final answer is $a+d+g=9+3+6=18$.

CC224. What is the smallest positive integer n such that 31 divides 5^n+n?

Originally Question 29 from 2003 High School Math Contest of University of South Carolina.

We received 12 correct solutions and we present the solution by Steven Chow.

Observe that $5^1\equiv 5 \pmod{31}$, $5^2\equiv 25 \pmod{31}$, and $5^3\equiv 1 \pmod{31}$, so for all integers $a\ge 0$,

$$5^a\equiv\begin{cases}1 \pmod{31}, & \text{if } a\equiv 0 \pmod{3},\\ 5 \pmod{31}, & \text{if } a\equiv 1 \pmod{3},\\ 25 \pmod{31}, & \text{if } a\equiv 2 \pmod{3}.\end{cases}$$

If $n\equiv 0 \pmod 3$, then $0\equiv 5^n+n\equiv 1+n \pmod{31} \iff n\equiv 30 \pmod{31}$, so the least possible n is 30.

If $n\equiv 1 \pmod 3$, then $0\equiv 5^n+n\equiv 5+n \pmod{31} \iff n\equiv 26 \pmod{31}$, so the least possible n is 88.

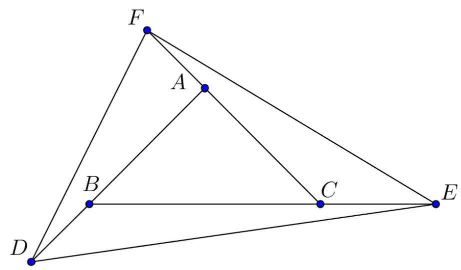
If $n\equiv 2 \pmod 3$, then $0\equiv 5^n+n\equiv 25+n \pmod{31} \iff n\equiv 6 \pmod{31}$, so the least possible n is 68.

Therefore the least possible value for n is 30.

Editor's Comments. David Manes used Fermat's Little Theorem in order to prove that $5^{30}+30\equiv 0 \pmod{31}$ and proved that there are no solutions for $n<30$. Konstantine Zelator showed also that the set of all positive integers n such that 5^n+n is divisible by 31 is the union of the three disjoint sets

$$\begin{aligned}S_1 &= \{n\mid n=93t+30, t\in\mathbb{N}\},\\ S_2 &= \{n\mid n=93q+88, q\in\mathbb{N}\},\\ S_3 &= \{n\mid n=93u+68, u\in\mathbb{N}\}.\end{aligned}$$

CC225. The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{2}AB, CE = \frac{1}{2}BC$ and $AF = \frac{1}{2}CA$. What is the ratio of the area of triangle DEF to that of triangle ABC?



Originally Question 30 from 2003 High School Math Contest of University of South Carolina.

We received eleven correct and two incorrect solutions. We present two solutions.

Solution 1, by Ricard Peiró i Estruch, slightly modified by the editor. The triangles ABC and AFB have the same altitude through A with the second triangle having half the base length of the first. Thus

$$[AFB] = \frac{1}{2}[ABC].$$

The triangles AFB and BFD have the same altitude through F, again with the second triangle having half the base length of the first, yielding

$$[BFD] = \frac{1}{2}[AFB] = \frac{1}{4}[ABC].$$

Similarly we obtain $[BDC] = [ACE] = \frac{1}{2}[ABC]$ and $[CDE] = [AEF] = \frac{1}{4}[ABC]$. Summing up all the areas gives

$$[DEF] = [ABC] + 3\left(\frac{1}{2}[ABC]\right) + 3\left(\frac{1}{4}[ABC]\right) = \frac{13}{4}[ABC].$$

Solution 2, by Andrea Fanchini. We use barycentric coordinates with respect to A, B, and C. Then the points D, E, F have coordinates

$$D\left(-\frac{1}{2}, \frac{3}{2}, 0\right), \qquad E\left(0, -\frac{1}{2}, \frac{3}{2}\right), \qquad F\left(\frac{3}{2}, 0, -\frac{1}{2}\right).$$

Therefore

$$[DEF] = \frac{[ABC]}{8}\begin{vmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \end{vmatrix},$$

implying

$$\frac{[DEF]}{[ABC]} = \frac{13}{4}.$$