

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2016: 42(5), p. 196–198.

CC221. What is the smallest positive integer n such that if S is any set containing n or more integers, then there must be three integers in S whose sum is divisible by 3?

Originally Question 29 from 2001 High School Math Contest of University of South Carolina.

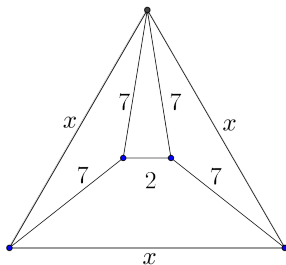
We received twelve solutions, all of which were correct.

If there are three integers that have a sum divisible by three, then either

- a) all three are congruent modulo 3, or
- b) their residues modulo 3 must be 0, 1, and 2.

It is possible to avoid either of these scenarios in a set of three integers; for example, the elements in $\{3, 4, 6\}$ have residues (mod 3) of $\{0, 1, 0\}$ or in a set of four integers (take $\{3, 4, 6, 7\}$, with residues (mod 3) of $\{0, 1, 0, 1\}$). A set of five integers must, by the pigeonhole principle, either have three elements with the same residue (mod 3) (case (a)) or three elements with residues (mod 3) of 0, 1, and 2 (case (b)). Thus the smallest set is five.

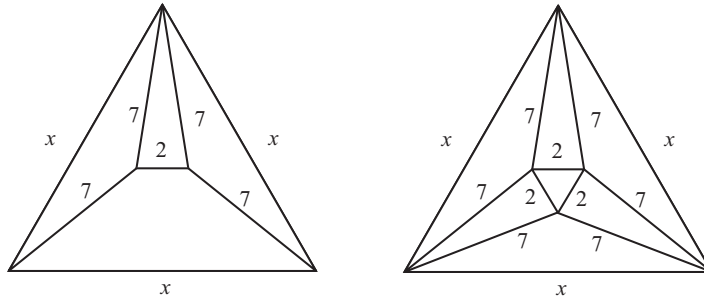
CC222. What is the value of x in the plane figure shown?



Originally Question 30 from 2002 High School Math Contest of University of South Carolina.

We received 14 correct and complete solutions, out of which we present the one by Ángel Plaza.

The figure below shows that by symmetry we can add other line segments and their known lengths.



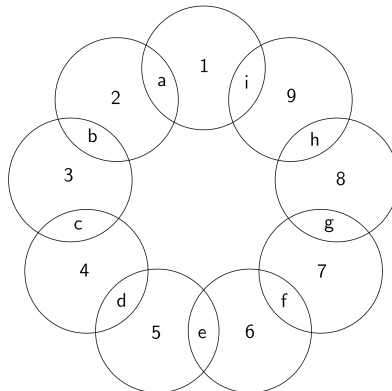
The area of the large equilateral triangle (in terms of x) may now be obtained in two ways, by directly calculating the area of the triangle, or by calculating the area of each of the sub-triangles. Thus

$$\frac{x^2\sqrt{3}}{2} = 3\left(\frac{x}{2}\sqrt{49 - \frac{x^2}{4}}\right) + 3(\sqrt{48}) + \sqrt{3}.$$

The only positive solution of this equation is $x = 13$.

Editor's comments. For an alternative solution inspired by the symmetry of the diagram on the right, note that in an equilateral triangle with sidelength x the distance from a corner to the center is $\frac{x}{\sqrt{3}}$, two thirds of the height. As the center of the large equilateral triangle is the center of the inner equilateral triangle as well, we can also evaluate this distance as $\sqrt{48} + \frac{1}{\sqrt{3}} = \frac{13}{\sqrt{3}}$, which immediately gives $x = 13$.

CC223. The letters a, b, c, d, e, f, g, h and i in the figure below represent the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 in a certain order. In each of the nine circles, we sum the three numbers so that nine sums are obtained. Suppose that all nine sums are equal. What is the value of $a + d + g$?



Originally Question 28 from 2003 High School Math Contest of University of South Carolina.

We received nine solutions, all of which were correct. We present an editor's amalgamation of the versions submitted by Somasundaram Muralidharan and Ricard Peiró i Estruch.

The sum of the numbers in the nine circles is

$$(1 + 2 + \cdots + 9) + 2 \cdot (a + b + \cdots + i) = 3 \cdot 45 = 135.$$

Since each circle contains the same sum, the sum in each circle must be $135/9 = 15$.

The circle holding the number 1 gives us $1 + a + i = 15$, so $a + i = 14$. Since every letter must take on an integer value between 1 and 9 inclusive, the only possible pairs for (a, i) are $\{(6, 8), (8, 6), (5, 9), (9, 5)\}$. The circle to the right gives us $i + 9 + h = 15$, so $i + h = 6$, which implies $i \leq 5$. The only possibility for the pair (a, i) is $(9, 5)$. Then $h = 1$.

We can now continue around the circuit clockwise, filling in one new letter as we consider each new circle. The final answer is $a + d + g = 9 + 3 + 6 = 18$.

CC224. What is the smallest positive integer n such that 31 divides $5^n + n$?

Originally Question 29 from 2003 High School Math Contest of University of South Carolina.

We received 12 correct solutions and we present the solution by Steven Chow.

Observe that $5^1 \equiv 5 \pmod{31}$, $5^2 \equiv 25 \pmod{31}$, and $5^3 \equiv 1 \pmod{31}$, so for all integers $a \geq 0$,

$$5^a \equiv \begin{cases} 1 \pmod{31}, & \text{if } a \equiv 0 \pmod{3}, \\ 5 \pmod{31}, & \text{if } a \equiv 1 \pmod{3}, \\ 25 \pmod{31}, & \text{if } a \equiv 2 \pmod{3}. \end{cases}$$

If $n \equiv 0 \pmod{3}$, then $0 \equiv 5^n + n \equiv 1 + n \pmod{31} \iff n \equiv 30 \pmod{31}$, so the least possible n is 30.

If $n \equiv 1 \pmod{3}$, then $0 \equiv 5^n + n \equiv 5 + n \pmod{31} \iff n \equiv 26 \pmod{31}$, so the least possible n is 88.

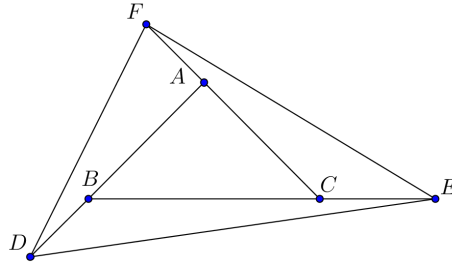
If $n \equiv 2 \pmod{3}$, then $0 \equiv 5^n + n \equiv 25 + n \pmod{31} \iff n \equiv 6 \pmod{31}$, so the least possible n is 68.

Therefore the least possible value for n is 30.

Editor's Comments. David Manes used Fermat's Little Theorem in order to prove that $5^{30} + 30 \equiv 0 \pmod{31}$ and proved that there are no solutions for $n < 30$. Konstantine Zelator showed also that the set of all positive integers n such that $5^n + n$ is divisible by 31 is the union of the three disjoint sets

$$\begin{aligned} S_1 &= \{n \mid n = 93t + 30, t \in \mathbb{N}\}, \\ S_2 &= \{n \mid n = 93q + 88, q \in \mathbb{N}\}, \\ S_3 &= \{n \mid n = 93u + 68, u \in \mathbb{N}\}. \end{aligned}$$

CC225. The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{2}AB$, $CE = \frac{1}{2}BC$ and $AF = \frac{1}{2}CA$. What is the ratio of the area of triangle DEF to that of triangle ABC ?



Originally Question 30 from 2003 High School Math Contest of University of South Carolina.

We received eleven correct and two incorrect solutions. We present two solutions.

Solution 1, by Ricard Peiró i Estruch, slightly modified by the editor. The triangles ABC and AFB have the same altitude through A with the second triangle having half the base length of the first. Thus

$$[AFB] = \frac{1}{2}[ABC].$$

The triangles AFB and BFD have the same altitude through F , again with the second triangle having half the base length of the first, yielding

$$[BFD] = \frac{1}{2}[AFB] = \frac{1}{4}[ABC].$$

Similarly we obtain $[BDC] = [ACE] = \frac{1}{2}[ABC]$ and $[CDE] = [AEF] = \frac{1}{4}[ABC]$. Summing up all the areas gives

$$[DEF] = [ABC] + 3\left(\frac{1}{2}[ABC]\right) + 3\left(\frac{1}{4}[ABC]\right) = \frac{13}{4}[ABC].$$

Solution 2, by Andrea Fanchini. We use barycentric coordinates with respect to A , B , and C . Then the points D, E, F have coordinates

$$D\left(-\frac{1}{2}, \frac{3}{2}, 0\right), \quad E\left(0, -\frac{1}{2}, \frac{3}{2}\right), \quad F\left(\frac{3}{2}, 0, -\frac{1}{2}\right).$$

Therefore

$$[DEF] = \frac{[ABC]}{8} \begin{vmatrix} -1 & 3 & 0 \\ 0 & -1 & 3 \\ 3 & 0 & -1 \end{vmatrix},$$

implying

$$\frac{[DEF]}{[ABC]} = \frac{13}{4}.$$