Characterizing a Symmedian

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Dedicated to the memory of Ross Honsberger

In 1873, Émile Lemoine introduced new cevians of the triangle, the symmedians, and presented some properties of their point of concurrency, a centre of the triangle now called the Lemoine point or symmedian point ([**1**]). Since then, these particular lines and point have been discussed in many geometry books and articles, as exemplified by the beautiful chapter 7 of Ross Honsberger's famous book [**2**] (one can also see [**3**], [**4**] or [**5**]). In this note, we examine several characterizations of the symmedian attached to a vertex of the triangle. We give unified proofs of some of these characterizations which are well-known and offer a couple of much less known ones.

Symmedians and antiparallels

First, let us recall the definition of a symmedian. Let m be the median through the vertex A of triangle ABC. The symmedian s through A is the reflection of the line m in the internal bisector ℓ of $\angle BAC$.

The median m and the symmedian s share a "bisection" property: clearly, m bisects any segment B_0C_0 with B_0 on AB, C_0 on AC and B_0C_0 parallel to BC, and therefore s bisects B_1C_1 where B_1, C_1 are the reflections in ℓ of B_0, C_0, respectively (Figure 1: the midpoints M_0 and M_1 of B_0C_0 and B_1C_1 are symmetric in ℓ).

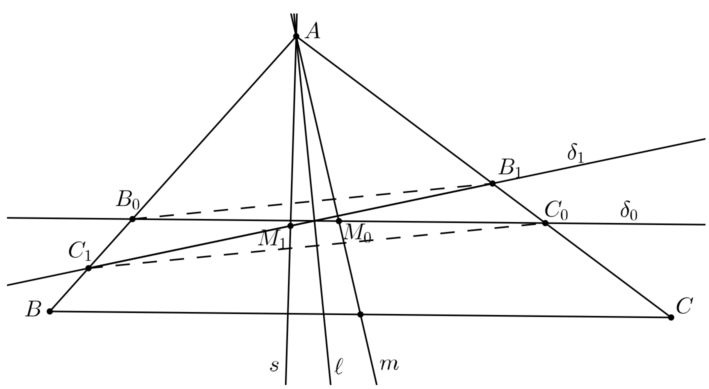

Figure 1

The segment B_1C_1 is said to be antiparallel to BC and the line $\delta_1 = B_1C_1$, the image of $\delta_0 = B_0C_0$ in ℓ, is called an antiparallel (line) to BC. With this terminology, the median m bisects any segment parallel to BC and the symmedian s bisects any segment antiparallel to BC. Since a reflection is involutive, we may even conclude:

> a line through A is the symmedian s if and only if it bisects some segment antiparallel to BC.

To emphasize this characterization, let us make two remarks. First, the antiparallels to BC can be recognized without involving ℓ explicitly. For example, they intersect AC, AB in B_1, C_1 such that ΔAB_1C_1 is inversely similar to ΔABC. More hidden is the following: the lines antiparallel to BC are exactly the perpendiculars to OA where O is the circumcentre of ΔABC. This easily follows from observing that a line δ_1 intersecting AB in C_1 and AC in B_1 is parallel to the tangent t at A to the circumcircle Γ of ΔABC if and only if $\angle(B_1A, B_1C_1) = \angle(BC, BA)$ (note that $\angle(BC, BA) = \angle(AC, t) = \angle(AB_1, t)$) (Figure 2).

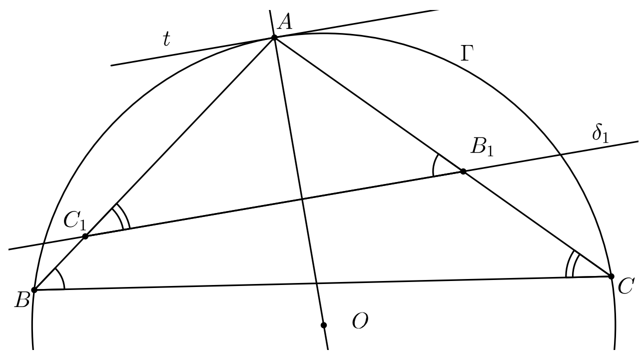

Figure 2

Our second remark is historical: in his original paper **[1]**, Lemoine *defined* the symmedians as the lines through each vertex bisecting any segment antiparallel to the opposite side, going so far as to call them *les médianes antiparallèles*. Perhaps the term *antimédianes* would have been a more appropriate choice!

Symmedians and polarity

A very well-known and often-used characterization of the symmedian s is the following one:

> s is the line through A and the pole of BC with respect to the circumcircle Γ of ΔABC.

In the proof (and coming proofs), we discard the easy cases when $\angle BAC = 90^\circ$ and when $AB = AC$ (in both cases, s is the altitude from A).

Recall that the pole P of BC with respect to Γ is the point of intersection of the tangents to Γ at B and C. Let BC meet the tangent t to Γ at A in Q and let AB and AC meet the tangent t' to Γ at the point A' diametrically opposite to A in B' and C', respectively. Let the line AP intersect BC at L and $B'C'$ at M (Figure 3). Since Q is on the polar BC of P and on the polar t of A, the line AP is the polar of Q. It follows that Q and L divide BC harmonically.

Under the central perspectivity with centre A, the points C, L, B, and Q are transformed into C', M, B', and the point at infinity on t', respectively, hence M is the midpoint of $B'C'$. Since $B'C'$ is antiparallel to BC, AP is the symmedian s.

In passing and for later use, note that the proof above readily yields another characterization associated with the polarity with respect to Γ:

> s is the polar of the intersection of BC with the tangent to Γ at A.

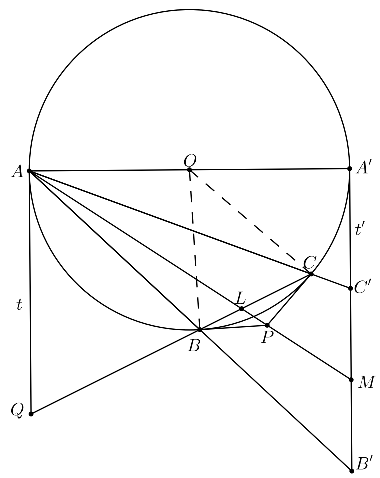

Figure 3

Symmedians and Grebe's construction

Another use of antiparallels leads to a proof of the following characterization of s:

> Let squares $ABDE$ and $ACFG$ be drawn externally to ΔABC and let R be the point of intersection of DE and FG. Then the line AR is the symmedian s.

This property provides a construction of the symmedian point known as Grebe's construction, from the German mathematician Ernst Grebe.

Let the lines AB and AC intersect RG and RE in B' and C', respectively (Figure 4). Clearly, $AB'RC'$ is a parallelogram so that RA bisects the segment $B'C'$. Therefore, we just have to prove that $B'C'$ is antiparallel to BC.

Let ρ denote the right-angle rotation with centre A transforming C into G (note that $\rho(E) = B$). Let $C_1 = \rho(C')$ and $B_1 = \rho^{-1}(B')$. Then, the line AB_1 is perpendicular to AB, hence parallel to BC_1 (note that $\angle ABC_1 = \angle AEC' = 90^\circ$ since $\rho(E) = B$ and $\rho(C') = C_1$). It follows that the line through the midpoints of AB and AC_1 is parallel to AB_1, hence intersects B_1C_1 at its midpoint and, being perpendicular to AB, is the perpendicular bisector of AB. In a similar way, the line through the midpoints of AC and AB_1 is the perpendicular bisector of AC and passes through the midpoint of B_1C_1. As a result, this midpoint is the circumcentre O of ABC.

Now, the vector $2\overrightarrow{AO} = \overrightarrow{AB_1} + \overrightarrow{AC_1}$ is the image under a right-angle rotation of $-\overrightarrow{AB'} + \overrightarrow{AC'} = \overrightarrow{B'C'}$, hence $B'C'$ is perpendicular to OA, and as such, is antiparallel to BC.

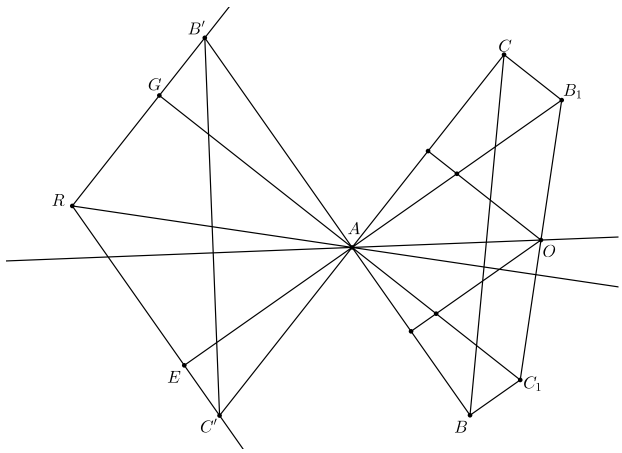

Figure 4

As a corollary, remarking that $\angle AGR = \angle AER = 90°$ so that the common midpoint of AR and $B'C'$ is equidistant from A, R, G, E, we obtain a characterization mentioned in **[5]** with a different proof:

> s is the line through A and the circumcentre of ΔAGE.

Symmedians as tangents

The next characterization is derived from an old problem proposed in 1928 in **[6]** and, to the best of my knowledge, does not appear in recent books or articles:

> s is the tangent at A to the circumcircle of $\Delta CAB'$ where B' is the reflection of B in A.

The proof given here uses the same method as above and is completely different from the 1928 solution by G. Excoffier.

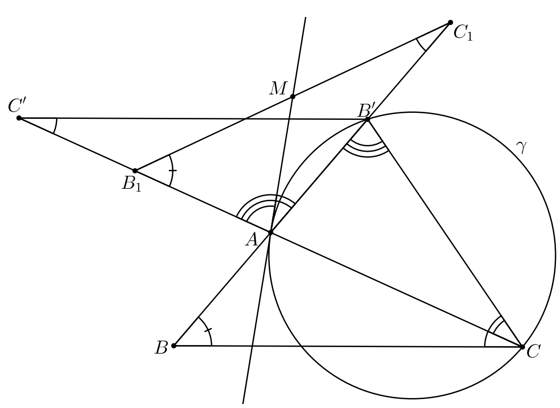

Figure 5

We introduce the symmetric C' of C about A, obtaining the parallelogram $CBC'B'$. Let B_1 and C_1 be the reflections of B' and C' in the internal bisector of $\angle BAC$, so that the segment B_1C_1 is antiparallel to BC. Let γ denote the circle through C, A, B' and let the tangent to γ at A intersect B_1C_1 at M (Figure 5). It suffices to show that $MB_1 = MC_1$.

We shall exploit the numerous equalities of angles of the figure: first

$$\angle AB_1C_1 = \angle CBA \quad \text{and} \quad \angle AC_1B_1 = \angle BCA = \angle B'C'C$$

(since B_1C_1 is antiparallel to BC and $B'C'$ is parallel to BC); second

$$\angle(CB', CA) = \angle(AC_1, AM) \quad \text{and} \quad \angle(B'A, B'C) = \angle(AM, AB_1)$$

(since AM is tangent to γ). We immediately deduce that the triangles AMC_1 and $CB'C'$ are similar and so are triangles AMB_1 and $B'CB$. In consequence, we have

$$MC_1 = AM \cdot \frac{B'C'}{CB'} \quad \text{and} \quad MB_1 = AM \cdot \frac{CB}{B'C},$$

and the equality $MB_1 = MC_1$ follows from $BC = B'C'$.

Symmedians and special circles through O

Our last characterization, which seems to be new, involves two particular circles passing through the circumcentre O of ΔABC:

> s is the line through the vertex A and the point of intersection other than O of the circumcircle of ΔBOC and the circle with diameter AO.

The proof, unlike the previous ones, leaves aside the antiparallels. Again we introduce the circumcircle Γ of ΔABC and recall that its tangent t at A intersects BC at the pole Q of the symmedian s.

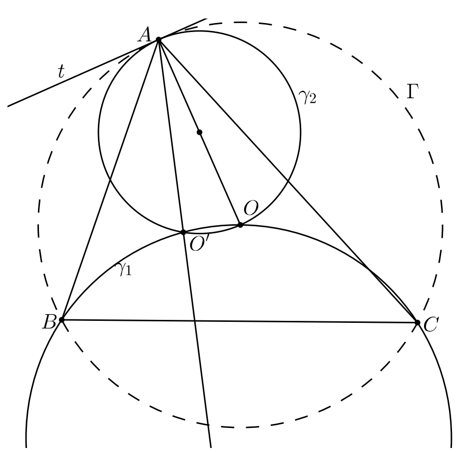

Figure 6

Let the circles γ_1 through B, O and C and γ_2 with diameter AO intersect at O and O' (Figure 6) and let $\mathbf{I}$ denote the inversion in the circle Γ. Then, $\mathbf{I}(\gamma_1)$ is the line BC and $\mathbf{I}(\gamma_2)$ is the tangent t so that $\mathbf{I}(O')$ is the point Q of intersection of t and BC. Since in addition AO' is perpendicular to OO', we conclude that AO' is the polar of Q with respect to Γ and the proof is complete.

References

[1] E. Lemoine, Note sur un point remarquable du plan d'un triangle, *Nouv. Ann. de Mathématiques*, tome 12, 1873, p. 364-6.
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[5] S. Luo and C. Pohoata, Let's Talk About Symmedians!, *Mathematical Reflections*, **4**, 2013.
[6] E. Mussel/G. Excoffier, Problème 10866, *Journal de mathématiques élémentaires*, Vuibert, 1928-9, No 1, p. 14.

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