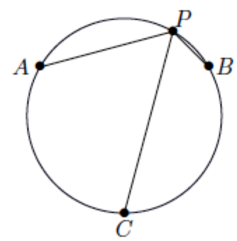
CC256. All vertices of a polygon P lie at points with integer coordinates in the plane (that is to say, both their co-ordinates are integers), and all sides of P have integer lengths. Prove that the perimeter of P must be even.

CC257. It is asserted that one can find a subset S of the nonnegative integers such that every nonnegative integer can be written uniquely in the form $x+2y$ for $x, y \in S$. Prove or disprove the assertion.

CC258. The three points A, B and C in the diagram are vertices of an equilateral triangle. Given any point P on the circle containing A, B and C, consider the three distances AP, BP and CP. Prove that the sum of the two shorter distances gives the longer distance.



CC259. If you are told that a rectangle has area A and perimeter P, is that sufficient information to determine its side lengths?

CC260. Assume you have a 9-faced die, appropriately constructed so that when the die is thrown, each of the faces (which are numbered 1 to 9) occurs with equal probability. Determine the probability that after n throws of the die, the product of all the numbers thrown will be divisible by 14.

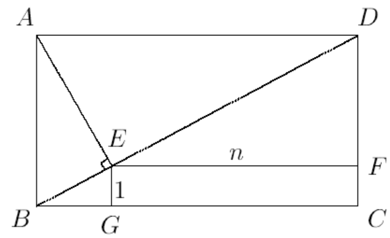
CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016: 42(2), p. 50–52.

CC206. A rectangle $ABCD$ has diagonal of length d. The line AE is drawn perpendicular to the diagonal BD. The sides of the rectangle $EFCG$ have lengths n and 1. Prove that $d^{2/3} = n^{2/3} + 1$.

Originally Question 2, Part B of the 1996 COMC paper (First Canadian Open Mathematics Challenge).

We received eleven correct solutions and present the solution by Ángel Plaza.



Let x be the length of BG. By the Pythagorean Theorem,

$$|BE| = \sqrt{1+x^2}, \quad |AE| = x\sqrt{1+x^2}, \quad |AB| = 1+x^2$$

since triangles ABE and BEG are similar. Therefore, $|DF| = x^2$ and $|EF| = n = x^3$ also because triangle ADF is similar to the previously considered triangles. By the Pythagorean Theorem,

$$|ED| = x^2\sqrt{1+x^2}, \quad \text{so} \quad d = |BE| + |ED| = (1+x^2)^{3/2}.$$

Then

$$d^{2/3} = 1 + x^2 = n^{2/3} + 1,$$

proving the claim.

CC207. Consider the ten numbers $ar, ar^2, \ldots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6, determine their product.

Originally Question 2, Part B of the 1997 COMC Paper.

We received ten correct solutions and one incorrect solution. We present the solution of Missouri State Problem Solving Group.

By the geometric series, the sum of the ten numbers is

$$\sum_{k=1}^{10} ar^k = ar\frac{1-r^{10}}{1-r}.$$

The sum of their reciprocals is also geometric, hence

$$\sum_{k=1}^{10} \frac{1}{ar^k} = \frac{1}{ar}\frac{1-r^{-10}}{1-r^{-1}} = \frac{1}{ar}\frac{r^{10}-1}{r^{10}-r^9} = \frac{1}{ar^{10}}\frac{1-r^{10}}{1-r}.$$

Therefore,

$$3 = 18 \div 6 = \left(\sum_{k=1}^{10} ar^k\right) \div \left(\sum_{k=1}^{10} \frac{1}{ar^k}\right) = ar\frac{1-r^{10}}{1-r} ar^{10}\frac{1-r}{1-r^{10}} = a^2r^{11}.$$

It then follows that the product of the ten numbers is

$$\prod_{k=1}^{10} ar^k = a^{10}r^{1+2+\cdots+10} = a^{10}r^{55} = \left(a^2r^{11}\right)^5 = 3^5 = 243.$$

CC208.

a) Let A and B be digits (that is, A and B are integers between 0 and 9 inclusive). If the product of the three-digit integers $2A5$ and $13B$ is divisible by 36, determine with justification the *four* possible ordered pairs (A, B).

b) An integer n is said to be a multiple of 7 if $n = 7k$ for some integer k.

   i) If a and b are integers and $10a + b = 7m$ for some integer m, prove that $a - 2b$ is a multiple of 7.

   ii) If c and d are integers and $5c + 4d$ is a multiple of 7, prove that $4c - d$ is also a multiple of 7.

Originally Question 2, part B of the 2002 COMC Paper.

We received eight solutions. We present the solution by Titu Zvonaru.

a) Since $36 = 4 \cdot 9$ and the integer $2A5$ is odd, then $13B$ is divisible by 4, hence $B = 2$ or $B = 6$. If $B = 2$, then 132 is divisible by 3. We deduce that $2A5$ is divisible by 3, hence $A = 2, 5, 8$. If $B = 6$, then $2A5$ is divisible by 9, hence $A = 2$. It follows that $(A, B) \in \{(2, 2), (5, 2), (8, 2), (2, 6)\}$.

b) If $10a + b$ is a multiple of 7, then

$$5(10a + b) = 49a + 7b + a - 2b = 7(7a + b) + (a - 2b)$$

is a multiple of 7, hence $a - 2b$ is a multiple of 7.

If $5c + 4d$ is a multiple of 7, then

$$5(5c + 4d) = 21c + 21d + 4c - d = 7(3c + 3d) + (4c - d)$$

is a multiple of 7, hence $4c - d$ is a multiple of 7.

CC209.

a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.

b) Determine the *one* value of x such that $x - \sqrt{4x + 12} = 0$. Justify your answer.

c) Determine all real values of c such that

$$x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$$

has precisely two distinct real solutions for x.

Originally Question 2, Part B of the 2004 COMC.

We received five correct solutions. We present the solution of Ángel Plaza.

a) $x = 6$ and $x = -2$ are the two solution of $x^2 - 4x - 12 = 0$. This can be seen through factoring or by the quadratic formula.

b) If $x - \sqrt{4x+12} = 0$ then $x = \sqrt{4x+12}$ and $x^2 = 4x + 12$, or rather $x^2 - 4x - 12 = 0$. So either $x = 6$ or $x = -2$. Checking we find $x = 6$ is the *one* value of x such that $x - \sqrt{4x+12} = 0$.

c) Notice that if $P(x) = x^2 - 4x - c$, then the solutions of the equation $P(x) - \sqrt{8P(x)} = 0$ are between the solutions of the equation $(P(x))^2 = 8P(x)$, that is $P(x)\,(P(x) - 8) = 0$.

Let Δ the discriminant of $P(x)$, that is $\Delta = 16+4c$, and Δ^* the discriminant of $P(x)-8$, that is $\Delta^* = 16+4c+32 = 4(12+c)$. Then $P(x)$ has two different real roots if and only if $c > -4$, one double real root if $c = -4$ and two not real roots if $c < -4$. Analogously, $P(x) - 8$ has two different real roots if and only if $c > -12$, one double real root if $c = -12$ and two non-real roots if $c < -12$. Consequently, equation $P(x)\,(P(x) - 8) = 0$ will have two real roots if and only if $c \in (-12, -4)$.

CC210. There is a unique triplet of positive integers (a, b, c) such that $a \le b \le c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

Originally Question 2, Part B of the 2013 COMC paper.

We received 13 solutions, of which ten were correct and complete. We present the solution by Steven Chow slightly modified by the editor.

Since $a \le b \le c$, then

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \le \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} \implies a \le 4.$$

Furthermore,

$$\frac{25}{84} > \frac{1}{a} \implies a \ge 4.$$

It follows that $a = 4$. We have

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \implies \frac{25}{21} = 1 + \frac{1}{b} + \frac{1}{bc} \implies (4b - 21)c = 21.$$

It follows that $4b - 21 > 0$, so $b \ge 6$, which yields $c \ge 6$. Since c divides 21, an easy check gives $b = 6$ and $c = 7$. Therefore $a + b + c = 17$.

Editor's Comments. Konstantine Zelator proved also that without the condition $a \le b \le c$, the only positive integer solutions to the equation

$$\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} = \frac{25}{84}$$

are $(a, b, c) = (4, 6, 7)$ and $(a, b, c) = (7, 1, 12)$.