SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2016: 42(2), p. 75–78.

4111. *Proposed by Mihaela Berindeanu.*

The circumscribed circle of $\triangle ABC$ has circumcenter O and circumradius R. Let P be a point on the side BC. Calculate R given that the maximum value of the product $PA \cdot PB \cdot PC$ is 2016.

We received five solutions, all correct, and will feature two of them.

Solution 1, by Roy Barbara.

We provide some clarification:

- If 2016 is the maximum of the product $PA \cdot PB \cdot PC$ for a *given* triangle ABC with P on the side labeled BC, then all we can say is that $R \geq 3\sqrt[3]{63}$.
- If 2016 is the maximum of the product $PA \cdot PB \cdot PC$ among all triangles ABC that are inscribed in a circle whose radius is R, then $R = 3\sqrt[3]{63}$.

Indeed, let Γ be a circle of radius R. Then, according to problem 10282 (proposed by Paul Erdős) in the *American Mathematical Monthly* **102**:5 (May 1995) 468-469, if P is a point lying interior to, or on a side of a triangle ABC that is inscribed in Γ, then

$$PA \cdot PB \cdot PC \leq \frac{32}{27}R^3,$$

and this maximum can be reached only for a point that lies on a side of the triangle at distance $\frac{R}{3}$ from the center. Now if $\frac{32}{27}R^3 = 2016$, then $R = 3\sqrt[3]{63}$.

Solution 2, by C.R. Pranesachar.

Suppose that we are given two points B and C on a circle with center O and radius R. Let P be a point of the chord BC and let $y = OP$. Then by a theorem of Euclid, the product $PB \cdot PC$ depends only on y and R; specifically,

$$PB \cdot PC = R^2 - y^2.$$

Further, it is known (and easily shown) that as a point A moves about the circumference, the distance AP achieves its maximum value of $AP = R + y$ when A is the farthest point from P on the diameter through P. For this position of A, one has

$$PA \cdot PB \cdot PC = (R + y)(R^2 - y^2).$$

If we denote the product on the right by $f(y)$ and solve $f'(y) = 0$ for y, we get $y = \frac{R}{3}$. Because $f''\left(\frac{R}{3}\right) < 0$, we conclude that $f(y)$ reaches its maximum in the interval $[0, R]$ at $y = \frac{R}{3}$, and this maximum value is $f\left(\frac{R}{3}\right) = \frac{32}{27}R^3$. Equating this to $2016 = 32 \times 63$, one gets $R = 3\sqrt[3]{63}$. We can construct a triangle that achieves this maximum value by taking a point P at a distance of $\frac{R}{3}$ from O, A as described above, and BC any chord through P such that B and C are distinct from A.

Editor's Comments. A version of the problem appeared earlier in *Crux* as problem 1895 proposed by Ji Chen and Gang Yu [1993: 295; 1994: 263; 1995: 204]. Because our problem 1895 came out around the same time as Erdős' *Monthly* version mentioned in solution 1 above, *Crux* provided a reference to the two "short and attractive" solutions that the *Monthly* published, and published none of its own. Both sources reported that the problem also appears in *Bull. Math. (Wuhan)*, 1990, No. 3 (sum No. 224), p. 17, with solution in 1991, No. 10 (sum No. 243) p. 42. Our solution 2 is similar to one of the solutions published in the *Monthly.*

4112. *Proposed by Ardak Mirzakhmedov and Leonard Giugiuc.*

Let ABC be an acute triangle. Prove that

$$\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2 B + 25} + \sqrt{96\cos^2 C + 25} \geq 21.$$

We received two submissions both of which are correct. We present the solution by Sefket Arslanagić, modified and enhanced by the editor.

We shall prove the given inequality under the relaxed condition that $A, B, C \in [0, \frac{\pi}{2}]$ such that $A + B + C = \pi$.
We first consider the special case when two of the three angles are equal; $A = B$ say. Since $C = \pi - 2A$, it suffices to prove that

$$2\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2(\pi - 2A) + 25} \geq 21$$

$$\text{or}\quad \sqrt{96\cos^2 2A + 25} \geq 21 - 2\sqrt{96\cos^2 A + 25} \qquad (1)$$

Clearly (1) holds if its right side is negative. Hence, we may assume that

$$2\sqrt{96\cos^2 A + 25} \leq 21.$$

Squaring both sides of (1) and using $2\cos^2 A = 1 + \cos 2A$ we obtain the following equivalent equations:

$$96\cos^2 2A + 25 \geq 441 - 84\sqrt{96\cos^2 A + 25} + 4(25 + 96\cos^2 A)$$

$$96\cos^2 2A + 25 \geq 441 - 84\sqrt{48(1 + \cos 2A) + 25} + 4(25 + 48(1 + \cos 2A))$$

$$84\sqrt{73 + 48\cos 2A} \geq 708 + 192\cos 2A - 96\cos^2 2A$$

$$7\sqrt{73 + 48\cos 2A} \geq 59 + 16\cos 2A - 8\cos^2 2A. \qquad (2)$$

Let $t = 2\cos 2A$. Then $t \in [-2, 2]$ and (2) becomes

$$7\sqrt{73 + t} \geq 59 + 8t - 2t^2. \qquad (3)$$

Squaring (3) and simplifying, we obtain the following equations:

$$\begin{aligned} &49(73+t) \geq 3481 + 944t - 172t^2 - 32t^3 + 4t^4 \\ &t^4 - 8t^3 - 43t^2 - 58t - 24 \leq 0 \\ &(t+1)^2(t-12)(t+2) \leq 0. \end{aligned} \tag{4}$$

Inequality (4) is true since $t \in [-2, 2]$, and the proof of (1) is complete. Therefore, we have shown that the given inequality holds if two of the angles are equal. Note that equality holds in (4) if and only if $t = -1$ or $t = -2$.

Now, $t = -1 \Rightarrow \cos 2A = -\frac{1}{2} \Rightarrow 2A = \frac{2\pi}{3} \Rightarrow A = \frac{\pi}{3}$, so $A = B = \frac{\pi}{3}$.

Also, $t = -2 \Rightarrow \cos 2A = -1 \Rightarrow 2A = \pi \Rightarrow A = \frac{\pi}{2}$, so $A = B = \frac{\pi}{2}$ and $C = 0$.

We now apply Lagrange's multipliers method to show that the minimum of

$$\sqrt{96\cos^2 A + 25} + \sqrt{96\cos^2 B + 25} + \sqrt{96\cos^2 C + 25}$$

is attained on $[0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ when two of A, B, and C are equal.

Let $F = \sqrt{73 + 48\cos 2A} + \sqrt{73 + 48\cos 2B} + \sqrt{73 + 48\cos 2C} + 2(A + B + C - \pi)$ where $A, B, C \in [0, \frac{\pi}{2}]$.

Thus, we have the following derivatives:

$$\begin{aligned} \frac{\partial F}{\partial A} &= \frac{-48\sin 2A}{\sqrt{73 + 48\cos 2A}} + \lambda, & \frac{\partial F}{\partial B} &= \frac{-48\sin 2B}{\sqrt{73 + 48\cos 2B}} + \lambda, \\ \frac{\partial F}{\partial C} &= \frac{-48\sin 2C}{\sqrt{73 + 48\cos 2C}} + \lambda, & \frac{\partial F}{\partial \lambda} &= A + B + C - \pi. \end{aligned} \tag{5}$$

Setting $\frac{\partial F}{\partial A} = \frac{\partial F}{\partial B} = \frac{\partial F}{\partial C} = \frac{\partial F}{\partial \lambda} = 0$, we see that $\lambda \geq 0$ since $A, B, C \in [0, \frac{\pi}{2}]$.

Now, if $\lambda = 0$, then $\sin 2A = \sin 2B = \sin 2C = 0$ which together with $A + B + C = \pi$ imply that $(A, B, C) = (0, \frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, 0, \frac{\pi}{2})$, or $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$.

Thus, we now assume that $\lambda > 0$. Then from (5) we have

$$\left(\frac{48}{\lambda}\right)^2 = \frac{73 + 48\cos 2A}{\sin^2 2A} = \frac{73 + 48\cos 2A}{1 - \cos^2 2A} = \frac{73 + 48\cos 2B}{1 - \cos^2 2B} = \frac{73 + 48\cos 2C}{1 - \cos^2 2C}.$$

Let $f(x) = \frac{73+48x}{1-x^2}$ where $x \in (-1, 1)$. Then

$$f'(x) = \frac{48(1 - x^2) + 2t(73 + 48x)}{(1 - x^2)^2} = \frac{48x^2 + 146x + 48}{(1 - x^2)^2} = \frac{2(8x + 3)(3x + 8)}{(1 - x^2)^2}.$$

Hence, $f'(x) < 0$ on $I_1 = \left(-1, -\frac{3}{8}\right)$ and $f'(x) > 0$ on $I_2 = \left(-\frac{3}{8}, 1\right)$. Note that two of $\cos 2A$, $\cos 2B$, and $\cos 2C$ must belong to the same I_n. So, assume $\cos 2A, \cos 2B \in I_1$. Then since $f(\cos 2A) = f(\cos 2B) = f(\cos 2C)$ and since

f is injective on each of I_1 and I_2, we then have $\cos 2A = \cos 2B$. Now since $2A, 2B \in [0, \pi]$, it follows that $2A = 2B$ and thus $A = B$.

Using this together with the result established in the first half of the solution, the proof is complete.

4113. *Proposed by Dragoljub Milošević.*

Let m_a, m_b and m_c be the lengths of medians, w_a, w_b and w_c be the lengths of the angle bisectors, r and R be the inradius and the circumradius, respectively, of a triangle. Prove that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \frac{1}{2} + \frac{r}{R} + \frac{R}{r}.$$

We received three correct solutions and present the solution by Šefket Arslanagić.

We have by the Cauchy-Buniakowski-Schwarz inequality

$$\begin{aligned}
\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} &= (m_a\sqrt{a})\left(\frac{1}{w_a\sqrt{a}}\right) + (m_b\sqrt{b})\left(\frac{1}{w_b\sqrt{b}}\right) + (m_c\sqrt{c})\left(\frac{1}{w_c\sqrt{c}}\right) \\
&\le \sqrt{(am_a^2 + bm_b^2 + cm_c^2)\left(\frac{1}{aw_a^2} + \frac{1}{bw_b^2} + \frac{1}{cw_c^2}\right)} \\
&= \sqrt{(am_a^2 + bm_b^2 + cm_c^2) \cdot \frac{1}{2F}\left(\frac{h_a}{w_a^2} + \frac{h_b}{w_b^2} + \frac{h_c}{w_c^2}\right)}. \qquad (1)
\end{aligned}$$

From p. 211, equation 10.5 of [2],

$$am_a^2 + bm_b^2 + cm_c^2 = \frac{s}{2}(s^2 + 2Rr + 5r^2) \qquad (2)$$

and from p. 57, equation (5) of [1],

$$\frac{h_a}{w_a}^{2} + \frac{h_b}{w_b^2} + \frac{h_c}{w_c^2} = \frac{R + 2r}{2Rr}. \qquad (3)$$

It follows now from (1), (2), and (3) that

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \sqrt{\frac{s}{2}(s^2 + 2Rr + 5r^2) \cdot \frac{1}{2rs} \cdot \frac{R + 2r}{2Rr}},$$

i.e.,

$$\frac{m_a}{w_a} + \frac{m_b}{w_b} + \frac{m_c}{w_c} \le \sqrt{\frac{1}{8}(s^2 + 2Rr + 5r^2) \cdot \frac{R + 2r}{Rr^2}}.$$

We will prove that

$$\sqrt{\frac{1}{8}(s^2 + 2Rr + 5r^2) \cdot \frac{R + 2r}{Rr^2}} \le \frac{1}{2} + \frac{r}{R} + \frac{R}{r}. \qquad (4)$$

This is successively equivalent to

$$\begin{aligned}
&\frac{1}{8}(s^2+2Rr+5r^2)\cdot\frac{R+2r}{Rr^2}\leq\left(\frac{1}{2}+\frac{r}{R}+\frac{R}{r}\right)^2,\\
&\frac{R+2r}{8Rr^2}(s^2+2Rr+5r^2)\leq\frac{1}{4}+\frac{r^2}{R^2}+\frac{R^2}{r^2}+\frac{r}{R}+\frac{R}{r}+2,\\
&R(R+2r)(s^2+2Rr+5r^2)\leq 2R^2r^2+8r^4+8R^4+8Rr^3+8R^3r+16R^2r^2,\\
&R(R+2r)s^2+R(R+2r)(2Rr+5r^2)\leq 8R^4+8r^4+8R^3r+8Rr^3+18R^2r^2,\\
&R(R+2r)s^2\leq 8R^4+6R^3r+9R^2r^2-2Rr^3+8r^4,\\
&s^2\leq\frac{8R^4+6R^3+9R^2r^2-2Rr^3+8r^4}{R^2+2Rr}.
\end{aligned}$$

We will use inequality 5.8 on p. 50 of [3]:

$$s^2\leq 4R^2+4Rr+3r^2, \tag{5}$$

and we will prove that

$$4R^2+4Rr+3r^2\leq\frac{8R^4+6R^3+9R^2r^2-2Rr^3+8r^4}{R^2+2Rr}. \tag{6}$$

This is successively equivalent to

$$\begin{aligned}
&(R^2+2Rr)(4R^2+4Rr+3r^2)\leq 8R^4+6R^3+9R^2r^2-2Rr^3+8r^4,\\
&2R^4-3R^3r-R^2r^2-4Rr^3+4r^4\geq 0,\\
&2\left(\frac{R}{r}\right)^4-3\left(\frac{R}{r}\right)^3-\left(\frac{R}{r}\right)^2-4\left(\frac{R}{r}\right)+4\geq 0.
\end{aligned}$$

Using the substitution $t=\frac{R}{r}$, this becomes successively

$$\begin{aligned}
2t^4-3t^3-t^2-4t+4&\geq 0,\\
(t-2)(2t^3+t^2+t-2)&\geq 0,\\
(t-2)[2(t^3-1)+t^2+t]&\geq 0,
\end{aligned}$$

which is clearly true by Euler's inequality $R\geq 2r$. From (5) and (6), inequality (4) holds, completing the proof of the claimed inequality. Equality holds for $t=2$, i.e, for equilateral triangles.

References

[1] S. Arslanagić: Eine neue geometrische Ungleichung in Dreieck, *Wurzel* **47** no. 3-4 (2013) 56-76.

[2] D.S. Mitrinović, J. E, Pecarić, and V. Volenec, *Recent Advances in Geometric Inequalities*. Kluwer Academic Publishers, Dordrecht, Boston, London, 1989.

[3] O. Bottema, R. Z. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.

4114. *Proposed by Michel Bataille.*

In the plane, let BC be a given line segment. Find the locus of A such that the centroid G of the triangle ABC satisfies $\angle GAB = \angle GBC$ and $\angle GAC = \angle GCB$.

We received six submissions, all correct. We present the solution by C.R. Pranesachar.

Let AG meet BC in the midpoint D of BC. We first observe that only one of the problem's angle requirements is needed; specifically,

$$\angle GAB = \angle GBC \quad \text{if and only if} \quad \angle GAC = \angle GCB.$$

For, if $\angle GAB = \angle GBC = \angle GBD$, for instance, then triangles GBD and BAD are similar, whence one has $DB^2 = DG \cdot DA$. Since $DB = DC$, it follows that $DC^2 = DG \cdot DA$, which implies that triangles GCD and CAD are similar, and one obtains $\angle GCB = \angle GCD = \angle DAC = \angle GAC$, as desired. It is now easily seen that the locus of A is a circle minus two of its points: From the statement $DB^2 = DG \cdot DA$ one has $\frac{1}{4}BC^2 = \frac{1}{3}DA^2$, whence

$$DA = \frac{\sqrt{3}}{2}BC.$$

Thus, for a fixed segment BC, the vertex A of triangles satisfying $\angle GAB = \angle GBC$ lies on a circle with D as centre and $\frac{\sqrt{3}}{2}BC$ as radius. Conversely, since all claims are reversible, if A is any point of this circle not on BC (so that ABC is a proper triangle), we have $\angle GAB = \angle GBC$, which completes the proof.

Comment. It may be noted that $\frac{1}{4}BC^2 = \frac{1}{3}DA^2$ is equivalent to the relation $2a^2 = b^2 + c^2$ (since by Stewart's theorem, $4AD^2 = 2(b^2 + c^2) - a^2$, where $a = BC$, etc.).

Editor's Comments. Bataille observed that any triangle ABC obtained from a point A of the locus is what is variously called a root-mean-square, or self-median, or quasi-isosceles triangle (see J. Chris Fisher, "Recurring ***Crux*** Configurations", [2011: 304-307]). The above solution shows that the angle condition $\angle GAB = \angle GBC$ characterizes such triangles.

4115. *Proposed by Daniel Sitaru.*

Prove that for all natural numbers $n \geq 2$, we have

$$n^{\ln 2} \leq \sqrt[3]{3} \cdot \sqrt[n+1]{n} \cdot \sqrt[n+2]{n} \cdot \ldots \cdot \sqrt[2n]{n}.$$

We received 6 solutions and 2 incomplete submissions. We present the solution by the Missouri State University Problem Solving Group.

Since $3\ln 2 = \ln 8 < \ln 9 = 2\ln 3$, we have $\frac{\ln 2}{2} < \frac{\ln 3}{3}$. If $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1-\ln x}{x^2} < 0$ for all $x > e$, so $f(x)$ is decreasing for $x > e$. Together these two observations imply

$$\frac{\ln 3}{3} \geq \frac{\ln n}{n} \text{ for all natural } n \geq 2.$$

As an upper Riemann sum,

$$\sum_{k=n}^{2n} \frac{1}{k} > \int_n^{2n+1} \frac{1}{x}\,dx = \ln \frac{2n+1}{n} = \ln\left(2+\frac{1}{n}\right) > \ln 2.$$

Using these two inequalities we get

$$\begin{aligned}\ln\left(\sqrt[3]{3}\cdot \sqrt[n+1]{n}\cdot \sqrt[n+2]{n}\cdot\ldots\cdot \sqrt[2n]{n}\right) &= \frac{\ln 3}{3} + \frac{\ln n}{n+1} + \frac{\ln n}{n+2} + \ldots + \frac{\ln n}{2n}\\ &\geq \frac{\ln n}{n} + \frac{\ln n}{n+1} + \frac{\ln n}{n+2} + \ldots + \frac{\ln n}{2n}\\ &= \ln n \cdot \sum_{k=n}^{2n} \frac{1}{k}\\ &> \ln n \ln 2\\ &= \ln n^{\ln 2}.\end{aligned}$$

Exponentiation yields the desired result.

4116. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers such that $a+b+c=3$. Find the minimum value of the expression

$$a^2(a^2-a+1) + b^2(b^2-b+1) + c^2(c^2-c+1).$$

There were 23 correct solutions, with two from one submitter. Most of the solvers used the approach of one of the following two solutions, with 11 relying on Jensen's Inequality. One solver pointed out that the result holds even when a, b, c are not all positive. Two used Lagrange Multipliers, with Kayla Cowan and Amanda Fox, students at the Southeast Missouri State University at Cape Girardeau, MI giving a careful and comprehensive treatment. We present two composite solutions.

Solution 1.

The minimum value is 3. Using $x^2 - x + 1 \geq x$ for $x \geq 0$ and a power means inequality, we get that

$$\begin{aligned}a^2(a^2-a+1) + b^2(b^2-b+1) + c^2(c^2-c+1) &\geq a^3+b^3+c^3\\ &\geq 3\left(\frac{a+b+c}{3}\right)^3 = 3.\end{aligned}$$

Equality occurs iff $a=b=c=3$.

Solution 2.

Let $f(x) = x^2(x^2-x+1)$. Because $f''(x) = (3x-1)^2 + (3x^2+2) > 0$, $f(x)$ is a convex function, and we can apply Jensen's Inequality. Thus,

$$\begin{aligned}&\frac{1}{3}\left[a^2(a^2-a+1)+b^2(b^2-b+1)+c^2(c^2-c+1)\right]\\&=\frac{1}{3}[f(a)+f(b)+f(c)]\geq f\left(\frac{a+b+c}{3}\right)=f(1)\geq 1,\end{aligned}$$

with equality iff $a=b=c=1$. Thus the minimum value of $f(a)+f(b)+f(c)$ for real a,b,c is 3.

Editor's Comments. The inequalities required in Solution 1 were obtained in some interesting ways. We will note the key ideas and leave the reader to complete the details.

$$\begin{aligned}&\sum a^4+\sum a^2\geq 2\sqrt{\sum a^4\cdot\sum a^2}\geq 2\sum a^3;\\&3\sum a^2(a^2-a+1)\geq\sum a^2(a^2+a+1);\\&\sum a^3=3\sum\left(\frac{a^3+1+1}{3}-\frac{2}{3}\right)\geq 3\sum\left(a-\frac{2}{3}\right)=3;\end{aligned}$$

$$\begin{aligned}a^3+b^3+c^3&=a^2\cdot a+b^2\cdot b+c^2\cdot c\geq\frac{1}{3}(a^2+b^2+c^2)(a+b+c)\\&=a^2+b^2+c^2\geq\frac{1}{3}(a+b+c)^2=3.\end{aligned}$$

4117. *Proposed by Martin Lukarevski.*

The sequence (x_n) is given recursively by $x_0=0, x_1=1$,

$$x_{n+1}=x_n\sqrt{x_{n-1}^2+1}+x_{n-1}\sqrt{x_n^2+1},\quad n\geq 1.$$

Find x_n.

We received 15 solutions. We present the solution by Prithwijit De.

Obviously $x_n\geq 0$ for all $n\geq 0$. The function $f:[0,\infty)\to[0,\infty)$ defined by

$$f(x)=\sinh(x)=\frac{e^x-e^{-x}}{2}$$

is a bijection. Thus for every nonnegative integer n there exists a $\theta_n\in[0,\infty)$ such that $x_n=\sinh(\theta_n)$. Using the given values for x_0 and x_1 we calculate that $\theta_0=0$ and $\theta_1=\ln(1+\sqrt{2})$.

Substitute $x_n=\sinh(\theta_n)$ in the given recursion, and use the hyperbolic sine formulas

$$\begin{aligned}&\cosh^2(x)=1+\sinh^2(x),\text{ and}\\&\sinh(x+y)=\sinh(x)\cosh(y)+\cosh(x)\sinh(y)\end{aligned}$$

to obtain

$$\sinh(\theta_{n+1}) = \sinh(\theta_n)\cosh(\theta_{n-1}) + \sinh(\theta_{n-1})\cosh(\theta_n) = \sinh(\theta_n + \theta_{n-1}).$$

As sinh is one-one on $[0,\infty)$ it follows that for $n \geq 1$ we have $\theta_{n+1} = \theta_n + \theta_{n-1}$, and hence $\theta_n = \theta_1 F_n$ where F_n is the n^{th} Fibonacci number (recall that the Fibonacci numbers are defined by the recursion $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$).

Thus for $n \geq 0$, we have

$$x_n = \sinh(\ln(1+\sqrt{2})F_n) = \frac{e^{\ln(1+\sqrt{2})F_n} - e^{-\ln(1+\sqrt{2})F_n}}{2} = \frac{(\sqrt{2}+1)^{F_n} - (\sqrt{2}-1)^{F_n}}{2}.$$

4118. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $a \in (0, \frac{\pi}{2}], b \in [\frac{\pi}{2}, \pi)$ with $a + b = \pi$. Calculate $\int_a^b \dfrac{x}{\sin x}dx$.

We received 11 correct solutions and will feature the one by Šefket Arslanagić here.

Substitute $x = \pi - y$, and using the fact that $a + b = \pi$, we have

$$I = \int_a^b \frac{x}{\sin x}dx = \int_a^b \frac{\pi - x}{\sin x}dx.$$

Then

$$\begin{aligned} 2I &= \int_a^b \frac{x}{\sin x}dx + \int_a^b \frac{\pi - x}{\sin x}dx \\ &= \pi \int_a^b \frac{dx}{\sin x} = \pi \ln(\tan(x/2))|_a^{\pi-a} \\ &= \pi \ln\left(\tan\left(\frac{\pi - a}{2}\right) / \tan\left(\frac{a}{2}\right)\right). \end{aligned}$$

Hence

$$I = \frac{\pi}{2} \ln\left(\tan\left(\frac{\pi - a}{2}\right) / \tan\left(\frac{a}{2}\right)\right).$$

4119. *Proposed by Ovidiu Furdui.*

Let $m, n, p \in \mathbb{N}$, $m \neq n$, and let A and B be 2×2 matrices with complex entries for which $mAB - nBA = pI_2$. Prove that

$$(AB - BA)^2 = O_2.$$

We received four submissions, three of which were correct and complete. We present two of the solutions.

Solution 1, by Michel Bataille.

Let $U = AB - BA$. Since $\mathrm{tr}(U) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$, the characteristic polynomial of U is $\chi_U(x) = x^2 + \det(U)$. From the Cayley-Hamilton Theorem, we have

$$U^2 + (\det(U))I_2 = 0,$$

so

$$U^2 = -(\det(U))I_2 = O_2$$

if $\det(U) = 0$. Thus, it suffices to show that $\det(U) = 0$ necessarily holds.

For the purpose of a contradiction, assume that $\det(U) \neq 0$. From the characteristic polynomial of U, we can conclude that U has two distinct eigenvalues, namely the square roots $\lambda_0,\ -\lambda_0$ of the nonzero complex number $-\det(U)$. It follows that there exist two independent column vectors X_1, X_2 such that $UX_1 = \lambda_0 X_1,\ UX_2 = -\lambda_0 X_2$.

Now, from $AB = \frac{n}{m}BA + \frac{p}{m}I_2$, we deduce

$$\lambda_0 X_1 = UX_1 = ABX_1 - BAX_1 = \frac{n-m}{m}\,BAX_1 + \frac{p}{m}\,X_1$$

so that

$$BAX_1 = \frac{p - \lambda_0 m}{m-n}\,X_1.$$

Similarly, we obtain

$$BAX_2 = \frac{p + \lambda_0 m + p}{m-n}\,X_2.$$

It follows that $\frac{p-\lambda_0 m}{m-n}$ and $\frac{p+\lambda_0 m}{m-n}$ are the eigenvalues of BA.

Using the same argument with $BA = \frac{m}{n}AB - \frac{p}{n}I_2$ we also have

$$ABX_1 = \frac{p - \lambda_0 n}{m-n}\,X_1 \quad \text{and} \quad ABX_2 = \frac{p + \lambda_0 n}{m-n}\,X_2,$$

so that $\frac{p-\lambda_0 n}{m-n}$ and $\frac{p+\lambda_0 n}{m-n}$ are the eigenvalues of AB.

Since AB and BA have the same characteristic polynomial, the product of the respective eigenvalues must be the same, that is

$$\frac{\lambda_0 m - p}{n-m} \cdot \frac{\lambda_0 m + p}{m-n} = \frac{\lambda_0 n - p}{n-m} \cdot \frac{\lambda_0 n + p}{m-n}.$$

It follows that $\lambda_0^2 m^2 - p^2 = \lambda_0^2 n^2 - p^2$, implying $m = n$, in contradiction with the hypothesis.

Solution 2, by the proposer.

We have $m(AB - BA) = pI_2 + (n-m)BA$ and thus, by passing to determinants,

$$\begin{aligned} m^2 \det(AB - BA) &= det(pI_2 + (n-m)BA) \\ &= (n-m)^2 \det\left(\frac{p}{n-m}I_2 + BA\right) \\ &= p^2 + p(n-m)\mathrm{tr}(BA) + (n-m)^2 \det(BA). \end{aligned}$$

Repeating the argument with $n(AB - BA) = pI_2 + (n - m)BA$, we also get

$$n^2 \det(AB - BA) = p^2 + p(n - m)\mathrm{tr}(AB) + (n - m)^2 \det(AB).$$

Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ and $\det(AB) = \det(BA)$, we obtain, by subtracting the previous equalities,

$$(m^2 - n^2)\det(AB - BA) = 0,$$

implying $\det(AB - BA) = 0$. Using the Cayley-Hamilton Theorem, we conclude $(AB - BA)^2 = O_2$, finishing the proof.

4120. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Find the minimum value of the function $f : [1, 2] \mapsto \mathbb{R}$, where

$$f(x) = \sqrt{\frac{8 - 3x}{x}} + 2\sqrt{4x + 1} - \sqrt{4x^2 - 8x + 49}.$$

We received nine correct solutions and present the solution by AN-anduud Problem Solving Group.

We have

$$\begin{aligned} f(x) \geq 0 &\Leftrightarrow \sqrt{\frac{8 - 3x}{x}} + 2\sqrt{4x + 1} \geq \sqrt{4x^2 - 8x + 49} \\ &\Leftrightarrow \sqrt{8 - 3x} + 2\sqrt{4x^2 + x} \geq \sqrt{4x^3 - 8x^2 + 49x} \\ &\Leftrightarrow (x - 1)^2(2 - x)(x^3 - 8x^2 + 23x - 2) \geq 0 \\ &\Leftrightarrow (x - 1)^2(2 - x)[x(x - 4)^2 + 5x + 2(x - 1)] \geq 0. \end{aligned}$$

Since $f(1) = f(2) = 0$, the minimum value of $f(x)$ is 0.