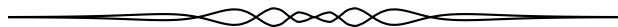


CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(9), p. 373–374, unless otherwise specified.



CC123. Find how many pairs of integers (x, y) satisfy the inequality

$$2^{x^2} + 2^{y^2} < 2^{1976}.$$

Originally question 5 on the 1976 entrance exam to the All-republican Distance Education Moscow School.

*This problem originally appeared in **CruX** 40(5), p. 188–189, but received no correct submissions. We since received two correct solutions. We present the solution by Billy Jin and Zi-Xia Wang.*

Note first that a necessary condition for $2^{x^2} + 2^{y^2} < 2^{1976}$ to hold is $x^2 < 1976$ and $y^2 < 1976$, or $|x| < \sqrt{1976}$ and $|y| < \sqrt{1976}$.

Since x and y are integers and $\lfloor \sqrt{1976} \rfloor = 44$, it follows that

$$x, y \in \{-44, -43, \dots, 0, 1, 2, \dots, 44\}. \quad (1)$$

Conversely, if (1) holds, then we have $1976 - x^2 > 1$ and $1976 - y^2 > 1$, which together imply that

$$\frac{2^{x^2} + 2^{y^2}}{2^{1976}} = \frac{1}{2^{1976-x^2}} + \frac{1}{2^{1976-y^2}} < \frac{1}{2} + \frac{1}{2} < 1,$$

so $2^{x^2} + 2^{y^2} < 2^{1976}$.

Therefore, since each of x and y can take 89 possible values, the total number of pairs (x, y) of integers satisfying $2^{x^2} + 2^{y^2} < 2^{1976}$ is $89^2 = 7921$.

CC125. Orthogonal projections of a triangle ABC onto two perpendicular planes are equilateral triangles with side length 1. If the median AD of triangle ABC has length $\sqrt{\frac{9}{8}}$, find BC .

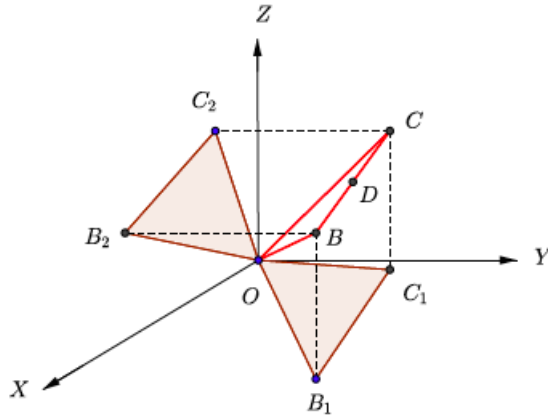
Originally question 4 on the 1969 entrance exam to the mathematical-mechanical department of Moscow State University.

*This problem originally appeared in **CruX** 40(5), p. 188–189, but received no correct submissions at that time. Here we present the solution by Somasundaram Muralidharan.*

Without loss of generality, we assume that the perpendicular planes on which the triangle is projected are the XOY and XOZ planes and that the vertex A of the

triangle is at the origin. Let the other two vertices have coordinates $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. The projections of these on the XOY plane are $B_1(b_1, b_2, 0)$ and $C_1(c_1, c_2, 0)$. Since $OB_1 = OC_1 = 1$ and $\angle B_1OC_1 = 60^\circ$, we have

$$b_1^2 + b_2^2 = 1, \quad c_1^2 + c_2^2 = 1, \quad b_1c_1 + b_2c_2 = \frac{1}{2}. \quad (2)$$



Similarly, the projections of B and C on the XOZ plane are $B_2(b_1, 0, b_3)$ and $C_2(c_1, 0, c_3)$. Using the fact that OB_2C_2 is an equilateral triangle with side equal to 1, we get

$$b_1^2 + b_3^2 = 1, \quad c_1^2 + c_3^2 = 1, \quad b_1c_1 + b_3c_3 = \frac{1}{2}. \quad (3)$$

Let D be the midpoint of BC . The length of the median $AD = \sqrt{\frac{9}{8}}$, so

$$\left(\frac{b_1 + c_1}{2}\right)^2 + \left(\frac{b_2 + c_2}{2}\right)^2 + \left(\frac{b_3 + c_3}{2}\right)^2 = \frac{9}{8} \quad (4)$$

$$(b_1^2 + b_2^2) + (c_1^2 + c_2^2) + 2(b_1c_1 + b_2c_2) + (b_3 + c_3)^2 = \frac{9}{2} \quad (5)$$

Using (2), we get $(b_3 + c_3)^2 = \frac{3}{2}$. Again, rewriting (4) as

$$(b_1^2 + b_3^2) + (c_1^2 + c_3^2) + 2(b_1c_1 + b_3c_3) + (b_2 + c_2)^2 = \frac{9}{2}$$

and using (3), we obtain $(b_2 + c_2)^2 = \frac{3}{2}$. Now from (4), we have $(b_1 + c_1)^2 = \frac{3}{2}$. Thus

$$b_1 + c_1 = \pm\sqrt{\frac{3}{2}}, \quad b_2 + c_2 = \pm\sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \pm\sqrt{\frac{3}{2}}$$

We need to discuss eight possibilities. We will consider two of these (the other cases are similar and can be mapped to one of these using rotation of axes).

Case 1. Suppose that

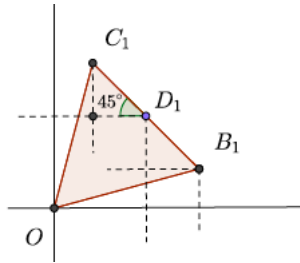
$$b_1 + c_1 = \sqrt{\frac{3}{2}}, \quad b_2 + c_2 = \sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \sqrt{\frac{3}{2}}.$$

Consider D_1 the projection of D on XOY plane. The coordinates of D_1 are

$$\left(\frac{b_1 + c_1}{2}, \frac{b_2 + c_2}{2}, 0\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}}, 0\right).$$

Thus D_1 lies on the line $y = x$ in the XOY plane and hence the coordinates of B_1 and C_1 are given by

$$(b_1, b_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right), (c_1, c_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).$$



If D_2 is the projection of D on the XOZ plane, then D_2 has coordinates

$$\left(\frac{b_1 + c_1}{2}, 0, \frac{b_3 + c_3}{2}\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, 0, \frac{1}{2}\sqrt{\frac{3}{2}}\right)$$

As before we find

$$(b_1, b_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right), (c_1, c_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).$$

Thus, in this case, we have

$$(b_1, b_2, b_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}\right),$$

$$(c_1, c_2, c_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}\right).$$

It readily follows that $BC = \sqrt{\frac{3}{2}}$.

Case 2. Suppose that

$$b_1 + c_1 = \sqrt{\frac{3}{2}}, \quad b_2 + c_2 = -\sqrt{\frac{3}{2}}, \quad b_3 + c_3 = \sqrt{\frac{3}{2}}.$$

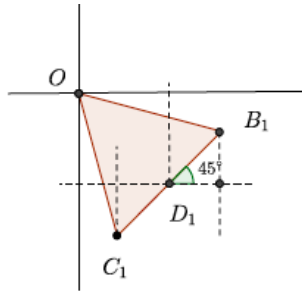
Consider D_1 the projection of D on XOY plane. The coordinates of D_1 are given by

$$\left(\frac{b_1 + c_1}{2}, \frac{b_2 + c_2}{2}, 0\right) = \left(\frac{1}{2}\sqrt{\frac{3}{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}}, 0\right).$$

In this case, D_1 lies on the line $y = -x$ and hence we find the coordinates of B_1, C_1 as (see the graphic)

$$(b_1, b_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}} \right),$$

$$(c_1, c_2) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}} \right).$$



As in **Case 1**, we find (b_3, c_3) as $b_3 = \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}$ and $c_3 = \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}$. Thus, in this case, we have

$$(b_1, b_2, b_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}} \right)$$

$$(c_1, c_2, c_3) = \left(\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, -\frac{1}{2}\sqrt{\frac{3}{2}} - \frac{1}{2\sqrt{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} + \frac{1}{2\sqrt{2}} \right)$$

Hence in this case also, we have $BC = \sqrt{\frac{3}{2}}$. This completes the proof.

CC152. A square of an $n \times n$ chessboard with $n \geq 5$ is coloured in black and white in such a way that three adjacent squares in either a line, a column or a diagonal are not all the same colour. Show that for any 3×3 square inside the chessboard, two of the squares in the corners are coloured white and the two others are coloured black.

Originally question 5 from 2013 Pan African Mathematics Olympiad.

*This problem originally appeared in **CruX** 41(1), p. 4-5, but received no correct submissions. We since received one correct solution by Maria Aleksandrova, presented below.*

Suppose there is a 3×3 square with three corners all the same colour. In our figures, we denote black by X and white by O .

$$\begin{array}{ccc} X & - & - \\ - & - & - \\ X & - & X \end{array}$$

In order for no square to line of 3 in a row to have the same colour, the rest of the grid must be filled in as follows:

$$\begin{array}{ccc} X & X & O \\ O & O & X \\ X & O & X \end{array}$$

We observe that the square beneath the right hand column cannot be X since this would give 3 in a row vertically, and it cannot be O , since this would give 3 in a row diagonally. Thus, this configuration must be on a side of the grid. Similarly, the square to the left of the top row cannot be filled. Thus, we see that the only place we can have 3 of the same colour in a 3×3 square is in a corner of the grid.

If we extend the 3×3 square up and to the right, we get:

$$\begin{array}{cccc} O & a & b & X \\ X & X & O & c \\ O & O & X & d \\ X & O & X & O \end{array}$$

Notice that c and d must both be different, since otherwise we would have 3 in a row vertically. If c is O and d is X then the square to the right of c could not be X or O , thus d is O and c is X . Similarly, a must be O and b must be X .

However, this gives a contradiction, as the diagonal containing a and d is monochromatic.

Thus, in any 3×3 square, two of the corners are each white and two are black.

CC191. There are 32 competitors in a tournament. No two of them are equal in playing strength, and in a one against one match the better one always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.

We received one complete and correct solution, by Somasundaram Muralidharan. An edited version is presented here.

In the first round the players are divided into 16 pairs, each of which plays a match, resulting in 16 winners. These 16 winners are divided into 8 pairs, each of which plays a second-round match, resulting in 8 winners. Four third-round matches give us 4 winners, then two fourth-round matches give us 2 winners, and one final match returns the gold medal winner. Finding the gold medal winner thus requires $16 + 8 + 4 + 2 + 1 = 31$ gold-medal matches over five rounds.

The silver medal winner must be one of the players who lost only to the gold medal winner in the gold-medal matches. There will be 5 such players, say p_1, p_2, p_3, p_4, p_5 , where p_i is the player who lost to the gold medal winner in the i^{th} round. The silver medal winner can be found among these 5 players over 4 silver-medal matches: p_1 plays p_2 , the winner plays p_3 , the winner of that match

plays p_4 , and the winner of that match plays p_5 . Finding the gold and the silver medal winners is thus accomplished in a total of $31 + 4 = 35$ matches.

The bronze medal winner must lose to only the gold and silver medal winners, so they must be among the players who lost to either the silver or gold medal winner in the gold- and silver-medal rounds. We claim that there will be at most 5 such players. Among those who lost to the gold medal winner, let the silver medal winner be p_j , where j indicates that they lost to the gold winner in the j^{th} gold-medal round. This means that they won $j - 1$ matches in the gold-medal stage. In the silver-medal stage they won either $5 - j + 1$ matches (if $j > 1$) or $5 - j$ matches if $j = 1$. Thus, over both the gold- and silver-medal stages, the silver medal winner won a maximum of 5 matches (the exception is 4 matches if $j = 1$), and thus beat a maximum of 5 players. The bronze medal winner can be determined from amongst these 5 players in 4 matches. Thus we are guaranteed to find the top three ranked players by playing a total of $31 + 4 + 4 = 39$ matches in all three stages.

We note that if we start with n players, determining the gold, silver and bronze medal winners requires a minimum of

$$\lceil \log_2(n(n-1)) \rceil + n - 3$$

matches, where $\lceil x \rceil$ is the smallest integer $\geq x$. For a proof, see Serge Tabachnikov (Editor), *Kvant Selecta: Combinatorics I*, American Mathematical Society, 2000.

CC192. Let M be a 3×3 matrix with all entries drawn randomly (and with equal probability) from $\{0, 1\}$. What is the probability that $\det M$ will be odd?

We received three correct and complete solutions, of which we present the one by the Missouri State University Problem Solving Group.

If we consider the entries to be from \mathbb{F}_2 , the field with two elements, the condition that $\det M$ be odd is equivalent to it being non-zero, which in turn is equivalent to M being invertible over \mathbb{F}_2 . This is equivalent to the rows of M being linearly independent. There are $2^3 - 1$ choices for the first row (we cannot choose the zero vector), $2^3 - 2$ choices for the second row (we cannot choose a scalar multiple of the first row), and $2^3 - 2^2$ choices for the third row (we cannot choose an element in the span of the first two rows). This gives $7 \cdot 6 \cdot 4 = 168$ invertible matrices out of a total of $2^9 = 512$ matrices, so our probability is $\frac{168}{512} = \frac{21}{64}$.

CC193. Consider the set of numbers $\{1, 2, \dots, 10\}$. Let $\{a_1, a_2, \dots, a_{10}\}$ be some permutation of these numbers and compute

$$|a_1 - a_2| + |a_3 - a_4| + \dots + |a_9 - a_{10}|.$$

What is the maximum possible value of the above sum over all possible permutations and how many permutations give you this maximum value?

We received two correct solutions. We present the solution of the Missouri State University Problem Solving Group.

We will solve the analogous problem when 10 is replaced by $2k$. To determine the maximum values, we assume without loss of generality that $a_{2i-1} > a_{2i}$. We then wish to maximize

$$\sum_{i=1}^k a_{2i-1} - \sum_{i=1}^k a_{2i}.$$

This will clearly occur when $\sum_{i=1}^k a_{2i-1}$ is maximized and $\sum_{i=1}^k a_{2i}$ is minimized, namely when

$$\{a_{2i-1}\}_{i=1}^k = \{k+1, k+2, \dots, 2k\}$$

and

$$\{a_{2i}\}_{i=1}^k = \{1, 2, \dots, k\}.$$

This gives a maximum value of

$$\sum_{i=k+1}^{2k} i - \sum_{i=1}^k i = \sum_{i=1}^k (k+i) - \sum_{i=1}^k i = \sum_{i=1}^k k + \sum_{i=1}^k i - \sum_{i=1}^k i = k^2.$$

If a permutation achieves this maximum, then for each of the pairs $\{a_{2i-1}, a_{2i}\}$, one element must be sent to an element of $A = \{1, 2, \dots, k\}$ and the other to an element of $B = \{k+1, k+2, \dots, 2k\}$. There are $k!$ ways of assigning elements of A to each i and $k!$ ways of assigning elements of B to each i . There are also 2 choices for each i , whether a_{2i-1} is sent to A or B (and hence a_{2i} is sent to B or A respectively). This gives a total of $2k(k!)^2$ permutations. For the original problem, the maximum value is $5^2 = 25$ and the number of permutations is $25(5!)^2 = 460800$.

CC194. At a strange party, each person knew exactly 22 others. For any pair of people X and Y who knew one another, there was no other person at the party that they both knew. For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew. How many people were at the party?

We received one complete and correct solution, by the Missouri State University Problem Solving Group; it is presented here.

Consider K_n the complete graph on n vertices (one vertex for each person at the party). Colour an edge blue if the people corresponding to its end-points know one another and red if they don't. We will count the number of triangles with two blue sides and one red one in two different ways. First, each person knows exactly 22 others and none of these 22 know each other. Therefore, if we choose two blue edges emanating from one vertex, the third side of the resulting triangle must be red and every triangle with exactly two blue edges arises in this manner. This gives $\binom{22}{2}n$ triangles with exactly two blue sides. Second, consider the subgraph consisting only of blue edges. It is well known that the number of edges of a graph is half the sum of the degrees. In this case, since the degree of a vertex of a blue edge is 22, the number of blue edges is $22n/2 = 11n$. The total number of edges in K_n is $\binom{n}{2}$. Therefore the number of red edges is $\binom{n}{2} - 11n$. Each red edge is an

edge in exactly 6 triangles with two blue edges, so the number of triangles with exactly two blue sides is $6 \binom{n}{2} - 11n$. Therefore,

$$\binom{22}{2}n = 6 \left(\binom{n}{2} - 11n \right).$$

Solving for n , we find $n = 0$, which we reject, or $n = 100$, so there are 100 people at the party.

CC195. A bisecting curve is one that divides a given region into two subregions of equal area. The shortest bisecting curve of a circle is clearly a diameter. What is the shortest bisecting curve of an equilateral triangle?

We received two submissions of which one was correct and complete. We present the solution by Somasundaram Muralidharan.

The shortest curve is an arc of a circle centred at one of the corners, with radius $\frac{3^{3/4}}{2\sqrt{\pi}}$ times the side length of the triangle.

Arrange six equilateral triangles to form a regular hexagon. Suppose that a curve dividing the area in half cuts two sides of the triangle. By piecing together six copies of the curve (see Figure 1), we obtain a curve that contains half the area of the hexagon. By the isoperimetric theorem, the shortest curve enclosing a given area is a circle. Thus, if we find the circle that encloses half the area of the hexagon, then the arc of that circle will also be the shortest bisecting curve for the equilateral triangle.

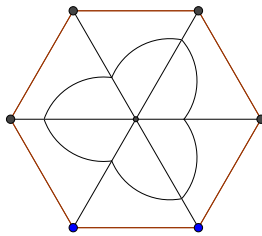


Figure 1: Curve bisecting the hexagon

Let the side of the equilateral triangle be 1. The area of the hexagon is $\frac{3\sqrt{3}}{2}$ and hence if r is the radius of the required circle, we have

$$\pi r^2 = \frac{3\sqrt{3}}{4} \Rightarrow r = \frac{3^{3/4}}{2\sqrt{\pi}}$$

Hence the length ℓ of the shortest arc bisecting the equilateral triangle is

$$\ell = \frac{2\pi r}{6} = \frac{\sqrt{\pi}}{2 \cdot 3^{1/4}}$$

Now suppose that there is a curve that bisects the area of the equilateral triangle and cuts only one of its sides. Piecing together two such curves, we obtain a curve enclosing an area equal to that of the triangle (See Figure 2). Again, we have a circle of radius r_1 which encloses the same area.

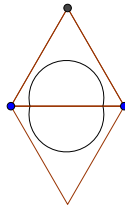


Figure 2: Circle bisecting the parallelogram

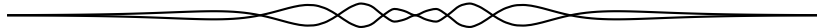
The radius r_1 is given by

$$\pi r_1^2 = \frac{\sqrt{3}}{4} \Rightarrow r_1 = \frac{3^{1/4}}{2\sqrt{\pi}}$$

The length ℓ_1 of the arc that bisects the equilateral triangle is given by

$$\ell_1 = \pi r_1 = \frac{\sqrt{\pi} \cdot 3^{1/4}}{2}$$

Clearly $\ell_1 > l$, and the proof is complete.



Editor's Note

Neculai Stanciu has brought it to our attention that the inequality discovered independently by A. Engel in 1998 and T. Andreescu in 2001 (see **CruX**, Volume 42 pages 216 and 342) was in fact known to H. Bergström as early as 1949.