SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(9), p. 397–400.

4081. *Proposed by Daniel Sitaru.*

Determine all $A, B \in M_2(\mathbb{R})$ such that:

$$\begin{cases} A^2 + B^2 = \begin{pmatrix} 22 & 44 \\ 14 & 28 \end{pmatrix}, \\ AB + BA = \begin{pmatrix} 10 & 20 \\ 2 & 4 \end{pmatrix}. \end{cases}$$

We received 17 correct solutions and will feature the solution by Joseph DiMuro.

Summing the two equations, we obtain:

$$(A+B)^2 = A^2 + AB + BA + B^2 = \begin{pmatrix} 32 & 64 \\ 16 & 32 \end{pmatrix}.$$

We can diagonalize this matrix in order to find its square roots:

$$(A+B)^2 = PDP^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix},$$

$$A + B = PD^{1/2}P^{-1} = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 8 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix} = \pm \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}.$$

We can also subtract the original two equations to obtain:

$$(A-B)^2 = A^2 - AB - BA + B^2 = \begin{pmatrix} 12 & 24 \\ 12 & 24 \end{pmatrix}.$$

As before, we diagonalize this matrix:

$$(A-B)^2 = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix},$$

$$A - B = PD^{1/2}P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \pm 6 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} = \pm \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

Now we have the two equations

$$A+B=\pm\begin{pmatrix}4 & 8\\ 2 & 4\end{pmatrix}, A-B=\pm\begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix},$$

which can easily be solved to produce four possible pairs of matrices for A and B. One solution is

$$A=\begin{pmatrix}3 & 6\\ 2 & 4\end{pmatrix}, B=\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}.$$

The other solutions may be obtained by interchanging A and B, and/or replacing A and B with their negatives.

4082. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let ABC be a right-angle triangle with $\angle A = 90^\circ$ and $BC = a$, $AC = b$ and $AB = c$. Consider the Fibonacci sequence F_n with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all non-negative integers n. Prove that

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq\frac{3}{2a^2}$$

for all non-negative integers m, n, p.

We received 8 correct solutions and present the solution by Adnan Ali.

From the Cauchy-Schwarz Inequality,
$(b^2+c^2)(F_k^2+F_\ell^2) = a^2(F_k^2+F_\ell^2) \geq (bF_k+cF_\ell)^2$, for all $k,\ell \geq 0$. Thus,

$$\frac{F_m^2}{(bF_n+cF_p)^2}+\frac{F_n^2}{(bF_p+cF_m)^2}+\frac{F_p^2}{(bF_m+cF_n)^2}\geq$$
$$\frac{F_m^2}{a^2(F_n^2+F_p^2)}+\frac{F_n^2}{a^2(F_p^2+F_m^2)}+\frac{F_p^2}{a^2(F_m+F_n^2)}\geq\frac{3}{2a^2},$$

where the last inequality follows from Nesbitt's Inequality. Equality holds iff $F_m = F_n = F_p$ and $b = c$.

Editor's Comments. As solvers pointed out, the fact that the F_n's were Fibonacci numbers was irrelevant; it was only necessary that they were nonnegative.

4083. *Proposed by Ovidiu Furdui.*

Calculate

$$\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\int_0^n\frac{x}{1+n\cos^2 x}\mathrm{d}x.$$

We received 10 solutions, of which 6 were correct and complete. We present the solution by Michel Bataille.

We show that the required limit is $\dfrac{1}{2}$.

Let $f_n(x) = \frac{1}{1+n\cos^2 x}$.

The π-periodicity of f_n and the change of variables $x = \tan^{-1}(t),\ dx = \frac{dt}{1+t^2}$ easily yield

$$\int_{(2k-1)\pi/2}^{(2k+1)\pi/2} f_n(x)\,dx = \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \int_{-\infty}^{\infty} \frac{dt}{n+1+t^2} = \frac{\pi}{\sqrt{n+1}}$$

for any $k, n \in \mathbb{N}$.

This said, for every $n \in \mathbb{N}$ with $n \geq 2$, let $p_n = \left\lfloor \frac{n}{\pi} + \frac{1}{2} \right\rfloor$ and $I_n = \int_0^n x f_n(x)\,dx$. Then, $(2p_n - 1)\frac{\pi}{2} \leq n < (2p_n + 1)\frac{\pi}{2}$ and

$$I_n = \int_0^{\pi/2} x f_n(x)\,dx + \sum_{k=1}^{p_n - 1} \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx + \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx.$$

Clearly,

$$0 \leq \int_0^{\pi/2} x f_n(x)\,dx \leq \frac{\pi}{2} \cdot \frac{1}{2} \int_{-\pi/2}^{\pi/2} f_n(x)\,dx = \frac{\pi^2}{4\sqrt{n+1}}$$

and for $k \in \{1, 2, \ldots, p_n - 1\}$,

$$(2k-1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}} \leq \int_{(2k-1)\pi/2}^{(2k+1)\pi/2} x f_n(x)\,dx \leq (2k+1)\frac{\pi}{2} \cdot \frac{\pi}{\sqrt{n+1}}.$$

Similarly,

$$0 \leq \int_{(2p_n-1)\frac{\pi}{2}}^{n} x f_n(x)\,dx \leq n \int_{(2p_n-1)\frac{\pi}{2}}^{n} f_n(x)\,dx \leq \frac{n\pi}{\sqrt{n+1}}.$$

Thus,

$$\frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k-1) \leq I_n \leq \frac{\pi^2}{4\sqrt{n+1}} + \frac{\pi^2}{2\sqrt{n+1}} \sum_{k=1}^{p_n-1} (2k+1) + \frac{n\pi}{\sqrt{n+1}}$$

so that

$$\frac{\pi^2 (p_n - 1)^2}{2\sqrt{n+1}} \leq I_n \leq \frac{\pi}{\sqrt{n+1}} \left(\frac{\pi}{4} + \frac{\pi}{2} \cdot p_n^2 + n \right) = \frac{\pi p_n^2}{\sqrt{n+1}} \left(\frac{\pi}{2} + \frac{\pi}{4p_n^2} + \frac{n}{p_n^2} \right).$$

Since $p_n \sim \frac{n}{\pi}$ as $n \to \infty$, we obtain

$$I_n \sim \frac{\pi^2 p_n^2}{2\sqrt{n+1}} \sim \frac{n\sqrt{n}}{2}$$

as $n \to \infty$. The result follows.

4084. *Proposed by Michel Bataille.*

In the plane, let Γ be a circle and A, B be two points not on Γ. Determine when $\frac{MA}{MB}$ is not independent of M on Γ and, in these cases, construct with ruler and compass I and S on Γ such that

$$\frac{IA}{IB} = \inf\left\{\frac{MA}{MB} : M \in \Gamma\right\} \quad \text{and} \quad \frac{SA}{SB} = \sup\left\{\frac{MA}{MB} : M \in \Gamma\right\}.$$

We feature the proposer's solution; we received no others.

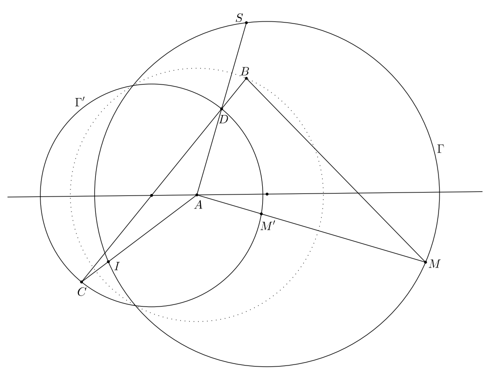
Because A is not on Γ, inversion in the circle with centre A and radius AB takes Γ to another circle, call it Γ'. For any point M on Γ, this inversion takes the pair of points M, B to another pair M', B, whose distances satisfy

$$MB = \frac{AB^2 \cdot M'B}{AM' \cdot AB} = \frac{AM \cdot AM' \cdot M'B}{AM' \cdot AB} = MA\frac{M'B}{AB};$$

consequently,

$$\frac{MA}{MB} = \frac{AB}{BM'}. \tag{1}$$

From (1), $\frac{MA}{MB}$ is independent of M on Γ if and only if BM' is constant. This occurs if and only if B is the centre of Γ'; that is, if and only if A and B are an inverse pair with respect to Γ. Otherwise, let the diameter of Γ' through B intersect Γ' at C and D with $BC > BD$. Then $\frac{MA}{MB}$ is minimal when BM' is maximal; that is, when $M' = C$; $\frac{MA}{MB}$ is maximal when BM' is minimal, in which case $M' = D$. Thus, I coincides with C' (the image of C under our inversion), and S coincides with D'. The construction of I and S is immediate once Γ' has been drawn. The circle Γ' can be readily constructed from the inverses of three points of Γ (as in the figure):



Editor's Comments. For any two fixed points A and B, the locus of points M for which $\frac{MA}{MB}$ is constant is called *the circle of Apollonius*; inversion in that circle interchanges A and B. See, for example H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited* (Mathematical Association of America, 1967), exercise 5.4.1, pages 114 and 172. Also, Theorem 5.41 there provides the distance formula used above to obtain (1).

4085. *Proposed by José Luis Díaz-Barrero. Correction.*

Let ABC be an acute triangle. Prove that

$$\sqrt[4]{\sin(\cos A)\cdot\cos B}+\sqrt[4]{\sin(\cos B)\cdot\cos C}+\sqrt[4]{\sin(\cos C)\cdot\cos A}<\frac{3\sqrt{2}}{2}.$$

We received eight submissions, six of which are correct. We present the solution by Titu Zvonaru.

It is well known that $\cos A+\cos B+\cos C\le\frac{3}{2}$ [Item 2.16 on p.22 of the book *Geometric Inequalities* by O. Bottema et al; Groningen, 1969]. Using this, together with the facts that $\sin x<x$ for $0<x<\frac{\pi}{2}$, $xy+yz+zx\le x^2+y^2+z^2$, and $(x+y+z)^2\le 3(x^2+y^2+z^2)$ we then have

$$\begin{aligned}\sum_{\text{cyc}}\sqrt[4]{\sin(\cos A)\cdot\cos B}&<\sum_{\text{cyc}}\sqrt[4]{\cos A\cdot\cos B}\le\sum_{\text{cyc}}\sqrt{\cos A}\\&\le\sqrt{3(\cos A+\cos B+\cos C)}\le\sqrt{3(\frac{3}{2})}=\frac{3\sqrt{3}}{2}.\end{aligned}$$

Editor's comments. Arkady Alt proved the stronger result that the given upper bound could be replaced by $3\sqrt[4]{\frac{1}{2}\sin\frac{1}{2}}$ which is less than $\frac{3\sqrt{3}}{2}$ since $\sin\frac{1}{2}<\frac{1}{2}$. This new upper bound is attained if and only if the triangle is equilateral. His proof used the Cauchy-Schwarz Inequality, concavity of the functions $\sqrt{\sin x}$ and $\sqrt{\cos x}$, Jensen's Inequality as well as the fact that $\sum\cos A=1+\frac{r}{R}$ and the Euler's Inequality $2r\le R$.

4086. *Proposed by Daniel Sitaru.*

Let be $f:[0,1]\to\mathbb{R}$; f twice differentiable on $[0,1]$ and $f''(x)<0$ for all $x\in[0,1]$. Prove that

$$25\int_{\frac{1}{5}}^{1}f(x)dx\ge 16\int_0^1 f(x)dx+4f(1).$$

We received seven solutions and present two of them.

Solution 1, by AN-anduud Problem Solving Group.

From the given conditions, f is concave on $[0,1]$. Using Hermite-Hadamard's inequality we get

$$\begin{aligned}16\int_{\frac{1}{5}}^{1}f(x)dx+9\int_{\frac{1}{5}}^{1}f(x)dx&\ge 16\cdot\int_{\frac{1}{5}}^{1}f(x)dx+9\cdot\frac{1-\frac{1}{5}}{2}\cdot\left(f(1)+f\left(\frac{1}{5}\right)\right)\\&=16\int_{\frac{1}{5}}^{1}f(x)dx+\frac{18}{5}f(1)+\frac{18}{5}f\left(\frac{1}{5}\right).\end{aligned}$$

On the other hand, we have

$$f\left(\frac{1}{5}\right)=f\left(\frac{1}{9}\cdot 1+\frac{8}{9}\cdot\frac{1}{10}\right)\ge\frac{1}{9}f(1)+\frac{8}{9}\cdot f\left(\frac{1}{10}\right),$$

so
$$\frac{18}{5} f\left(\frac{1}{5}\right) \geq \frac{2}{5} f(1) + \frac{16}{5} f\left(\frac{1}{10}\right).$$

From here, we get
$$25 \int_{\frac{1}{5}}^{1} f(x)dx \geq 16 \int_{\frac{1}{5}}^{1} f(x)dx + 4f(1) + \frac{16}{5} f\left(\frac{1}{10}\right).$$

Using Hermite-Hadamard's inequality, we get
$$f\left(\frac{1}{10}\right) = f\left(\frac{\frac{1}{5}+0}{2}\right) \geq \frac{1}{\frac{1}{5}-0} \int_0^{\frac{1}{5}} f(x)dx \iff \frac{1}{5} f\left(\frac{1}{10}\right) \geq \int_0^{\frac{1}{5}} f(x)dx.$$

Hence, we get
$$25 \int_{\frac{1}{5}}^{1} f(x)dx \geq 16 \int_0^1 f(x)dx + 4f(1).$$

Solution 2, by Leonard Giugiuc.

In $\displaystyle\int_{\frac{1}{5}}^{1} f(x)\ dx$, we make the substitution $x \to \dfrac{5x-1}{4}$ and clear fractions to get
$$25 \int_{\frac{1}{5}}^{1} f(x)\ dx = 20 \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx.$$

We need to prove
$$\begin{aligned} 20 \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx &\geq 16 \int_0^1 f(x)\ dx + 4f(1) \iff \\ \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx &\geq \frac{4}{5} \int_0^1 f(x)\ dx + \frac{1}{5} f(1) \iff \\ \int_0^1 f\left(\frac{4x+1}{5}\right)\ dx &\geq \int_0^1 \left[\frac{4}{5} f(x) + \frac{1}{5} f(1)\right]\ dx. \end{aligned}$$

But $f''(x) < 0\ \forall x \in [0,1]$, so f is concave on $[0,1]$ and from here
$$f\left(\frac{4x+1}{5}\right) \geq \frac{4}{5} f(x) + \frac{1}{5} f(1).$$

Integrating, we conclude that
$$\int_0^1 f\left(\frac{4x+1}{5}\right)\ dx \geq \int_0^1 \left[\frac{4}{5} f(x) + \frac{1}{5} f(1)\right]\ dx.$$

Editor's Comments. Henry Ricardo observed that this problem appears as problem MA 110 (with solution) in the Daniel Sitaru's book *Math Phenomenon*, published in English by the Romanian publisher Editura Paralela 45 in 2016.

4087. *Proposed by Lorian Saceanu.*

If S is the area of triangle ABC, prove that

$$m_a(b+c)+2m_a^2 \geq 4S\sin A,$$

where b and c are the lengths of sides that meet in vertex A, and m_a is the length of the median from that vertex; furthermore, equality holds if and only if $b=c$ and $\angle A = 120°$.

We received seven correct submissions and present the solution by Leonard Giugiuc.

Let A' be the reflection of A in the midpoint M of BC. Because $\Delta AMC \cong \Delta A'MB$, we have

$$AA' = 2m_a,\ A'B = b,\ \angle A'BA = \pi - A, \ \text{ and } \ [A'AB] = [ABC] = S$$

(where the square brackets denote area). Let $m = 2m_a$ and denote by r', R', and s' $(= \frac{m+b+c}{2})$ the inradius, circumradius, and semiperimeter, respectively, of $\Delta A'AB$. We need to prove that

$$m(b+c)+m^2 \geq 8S\sin\angle A'BA,$$

which is equivalent, in turn, to

$$\begin{aligned} m(m+b+c) &\geq 8S\sin(\pi - A) \\ \frac{2mS}{r'} &\geq 8S\sin A \\ \frac{m}{\sin A} &\geq 4r' \\ R' &\geq 2r'. \end{aligned}$$

But the final line is Euler's inequality applied to $\Delta A'AB$, which completes the proof. Equality holds for Euler's inequality if and only if $\Delta A'AB$ is equilateral, which implies that $b=c$ and $\angle A = 120°$, as desired.

4088. *Proposed by Ardak Mirzakhmedov.*

Let a, b and c be positive real numbers such that $a^2b + b^2c + c^2a + a^2b^2c^2 = 4$. Prove that

$$a^2+b^2+c^2+abc(a+b+c) \geq 2(ab+bc+ca).$$

We received four submissions all of which are correct. We present the solution by the proposer, expanded by the editor with some details.

We first show that the given condition implies

$$\frac{a}{2a+bc^2}+\frac{b}{2b+ca^2}+\frac{c}{2c+ab^2} = 1 \tag{1}$$

or

$$\begin{aligned} a(2b+ca^2)(2c+ab^2)+b(2c+ab^2)(2a+bc^2)+c(2a+bc^2)(2b+ca^2) \\ = (2a+bc^2)(2b+ca^2)(2c+ab^2). \end{aligned} \tag{2}$$

Let S and P denote the left side and the right side of (2), respectively. Then by straightforward computations, we find

$$\begin{aligned} S &= \sum_{\text{cyc}} a(4bc + 2ab^3 + 2c^2a^2 + a^3b^2c) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(a^3b + b^3c + c^3a) \\ &= 12abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + abc(4 - a^2b^2c^2) \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3 \end{aligned}$$

and

$$\begin{aligned} P &= (4ab + 2ca^3 + 2b^2c^2 + a^2bc^3)(2c + ab^2) \\ &= 8abc + 4(c^2a^3 + a^2b^3 + b^2c^3) + 2abc(a^3b + b^3c + c^3a) + a^3b^3c^3 \\ &= 8abc + 4(a^2b^3 + b^2c^3 + c^2a^3) + 2abc(4 - a^2b^2c^2) + a^3b^3c^3 \\ &= 16abc + 4(a^2b^3 + b^2c^3 + c^2a^3) - a^3b^3c^3. \end{aligned}$$

Hence, $S = P$ which establishes (2).

Now, for all $u, v, w > 0$, we have by the Cauchy-Schwarz Inequality that

$$(u + v + w)\left(\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w}\right) \geq (a + b + c)^2. \tag{3}$$

Setting

$$u = 2a^2 + bc^2a, \quad v = 2b^2 + ca^2b \quad \text{and} \quad w = 2c^2 + ab^2c,$$

we then have by (1) and (3) that

$$1 = \frac{a^2}{2a^2 + bc^2a} + \frac{b^2}{2b^2 + ca^2b} + \frac{c^2}{2c^2 + ab^2c} \geq \frac{(a + b + c)^2}{u + v + w},$$

so

$$(2a^2 + bc^2a) + (2b^2 + ca^2b) + (2c^2 + ab^2c) = u + v + w \geq (a + b + c)^2$$

from which it follows that

$$a^2 + b^2 + c^2 + abc(a + b + c) \geq 2(ab + bc + ca).$$

4089. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a, b, c and d be real numbers with $0 < a < b < c < d$. Prove that

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > 3 + \ln \frac{d}{a}.$$

There were 14 correct solutions. We present four of them here. Most of the solvers approached the problem along the lines of one of the first two solutions.

Solution 1.

Since $x > 1 + \ln x$ for $x \neq 1$,

$$\frac{b}{a} + \frac{c}{b} + \frac{d}{c} > \left(1 + \ln\frac{b}{a}\right) + \left(1 + \ln\frac{c}{b}\right) + \left(1 + \ln\frac{d}{c}\right) = 3 + \ln\frac{d}{a}$$

as desired.

Solution 2.

Applying the AM-GM Inequality, we find that the left side is not less than

$$3\sqrt[3]{\frac{d}{a}} > 3\left(1 + \ln\sqrt[3]{\frac{d}{a}}\right) = 3 + \ln\frac{d}{a}.$$

Solution 3, by Kee-Wai Lau.

For $0 < a < b < c < d$, let

$$\begin{aligned} f(a,b,c,d) &= \frac{b}{a} + \frac{c}{b} + \frac{d}{c} - \ln\frac{d}{a}, \\ g(a,b,c) &= \frac{b}{a} + \frac{c}{b} + 1 - \ln\frac{c}{a}, \\ h(a,b) &= \frac{b}{a} + 2 - \ln\frac{b}{a}. \end{aligned}$$

An analysis of the partial derivatives reveals that each of its functions strictly increases in its final variable, so that

$$f(a,b,c,d) > f(a,b,c,c) = g(a,b,c) > g(a,b,b) = h(a,b) > h(a,a) = 3,$$

which yields the desired result.

Solution 4, by the proposers.

Let $f(x) = 1/x$. A diagram shows that

$$(b-a)f(a) + (c-b)f(b) + (d-c)f(c) > \int_a^d \frac{dx}{x},$$

whence

$$\frac{b}{a} - 1 + \frac{c}{b} - 1 + \frac{d}{c} - 1 > \ln d - \ln a,$$

as desired.

Editor's Comments. Two solvers provided a straightforward generalization for an increasing sequence $\{a_k\}$ of $n+1$ positive reals:

$$\sum_{k=1}^{n} \frac{a_{k+1}}{a_k} > n + \ln\frac{a_{n+1}}{a_1}.$$

4090. *Proposed by Nermin Hodžić and Salem Malikić.*

Let a, b and c be non-negative real numbers such that $a^2+b^2+c^2=3$. Prove that
$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{3}{8}.$$

We received two correct solutions. We present the solution of the proposers, slightly modified by the editor.

Using Jensen's inequality for $f(x)=\frac{1}{x}$ (which is convex on $(0,\infty)$), we have
$$\frac{a}{a+b+c}\cdot\frac{1}{3b^2+6c-bc}+\frac{b}{a+b+c}\cdot\frac{1}{3c^2+6a-ca}+\frac{c}{a+b+c}\cdot\frac{1}{3a^2+6b-ab}$$
$$\geq\left(\frac{a(3b^2+6c-bc)}{a+b+c}+\frac{b(3c^2+6a-ca)}{a+b+c}+\frac{c(3a^2+6b-ab)}{a+b+c}\right)^{-1},$$
which we can rearrange to
$$\frac{a}{3b^2+6c-bc}+\frac{b}{3c^2+6a-ca}+\frac{c}{3a^2+6b-ab}\geq\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}.$$
In order to prove the inequality given in the question, it thus suffices to show
$$\frac{(a+b+c)^2}{3(ab^2+bc^2+ca^2)+6(ab+bc+ca)-3abc}\geq\frac{3}{8},$$
which holds (by cross multiplying and rearranging) if and only if
$$8(a+b+c)^2\geq 9(ab^2+bc^2+ca^2)+18(ab+bc+ca)-9abc \iff$$
$$8(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)+2(ab+bc+ca)-9abc.$$
By the Cauchy-Schwarz inequality, $ab+bc+ca\leq a^2+b^2+c^2$. Note for later that equality holds if and only if $a=b=c=1$. Hence, it suffices to show that
$$6(a^2+b^2+c^2)\geq 9(ab^2+bc^2+ca^2)-9abc.$$
Finally, since $a^2+b^2+c^2=3$, this reduces to
$$2\geq(ab^2+bc^2+ca^2)-abc. \tag{1}$$
Assume that $a\geq b\geq c$. Then $(a-b)(b-c)\geq 0$, equivalent to $ab+bc\geq b^2+ac$. Multiply both sides by $a>0$ and rearrange to get $abc\geq ab^2+a^2c-a^2b$. Note that the cubic $g(b)=3b-b^3$ has a local maximum at $b=1$, and in fact for all $b\geq 0$ we have $3b-b^3\leq g(1)=2$. Hence
$$abc+2\geq ab^2+a^2c-a^2b+3b-b^3,$$
which is equivalent to
$$abc+2\geq ab^2+a^2c+c^2b-b(a^2+b^2+c^2)+3b.$$
Since $a^2+b^2+c^2=3$, this shows that $abc+2\geq ab^2+a^2c+c^2b$, which is equivalent to (1), concluding the proof.