THE OLYMPIAD CORNER

No. 347

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC301. Solve the following Diophantine equation for integers x and y:

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that,

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

OC303. Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. Point P lies on AC such that $\angle BKP = 90^\circ$. Prove that $OP \parallel BC$.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer bigger than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

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OC301. Résoudre l'équation diophantienne

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3,$$

x et y étant des entiers.

OC302. Soit x, y et z des nombres réels dont les sommes deux à deux ne sont pas égales à 1. Démontrer que

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2}\geq 2(x+y+z)-\frac{3}{4}$$

et déterminer les triplets (x, y, z) qui vérifient l'égalité.

OC303. Soit ABC un triangle, H son orthocentre et O le centre du cercle circonscrit au triangle. Soit K le milieu de AH et P le point sur AC tel que $\angle BKP = 90°$. Démontrer que OP est parallèle à BC.

OC304. Soit k un entier fixe strictement positif. Soit $F(n)$ le plus petit entier strictement positif supérieur à kn tel que $F(n)\cdot n$ est un carré parfait. Démontrer que si $F(n) = F(m)$, alors $m = n$.

OC305. Soit p un nombre premier pour lequel $\frac{p-1}{2}$ est aussi un nombre premier et soit a, b et c des entiers qui ne sont pas divisibles par p. Démontrer qu'il existe au plus $1+\sqrt{2p}$ entiers strictement positifs n tels que $n < p$ et que p soit un diviseur de $a^n + b^n + c^n$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(7), p. 288–289.

OC241. Let n be a natural number. For every positive real numbers $x_1, x_2, ..., x_{n+1}$ such that $x_1x_2...x_{n+1} = 1$ prove that:

$$\sqrt[x_1]{n}+\cdots+\sqrt[x_{n+1}]{n}\geq n^{\sqrt[n]{x_1}}+\cdots+n^{\sqrt[n]{x_{n+1}}}.$$

Originally problem 5 from day 2 of the 2014 Iran Team Selection Test.

We received 2 correct submissions. We present the solution by Michel Bataille.

We shall use the following inequality of means: if $x, a_1, \ldots, a_n > 0$, then

$$\left(\frac{a_1^x+\cdots+a_n^x}{n}\right)^{1/x}\geq\sqrt[n]{a_1\cdots a_n}$$

which rewrites as

$$a_1^x+\cdots+a_n^x\geq n(a_1\cdots a_n)^{x/n}. \qquad (1)$$

Let

$$S = n^{1/x_1} + \cdots + n^{1/x_{n+1}},$$

$$p_j = \prod_{\substack{k=1 \\ k\neq j}}^{n} x_k = \frac{1}{x_{n+1}x_j}, \ (j = 1, 2, \ldots, n),$$

$$S_{n+1} = S - n^{1/x_{n+1}} = n^{1/x_1} + \cdots + n^{1/x_n}.$$

Using (1), we obtain

$$S_{n+1} = \sum_{k=1}^{n} (n^{p_j})^{x_{n+1}} \geq n \left(n^{p_1+\cdots+p_n}\right)^{\frac{x_{n+1}}{n}} = n \left(n^{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}\right)^{\frac{1}{n}}. \qquad (2)$$

Again by (1) with $x = 1$, that is, by AM-GM, we also have

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} \geq n \left(\frac{1}{x_1 \cdots x_n}\right)^{\frac{1}{n}} = n(x_{n+1})^{\frac{1}{n}}.$$

It follows that $n^{\frac{1}{x_1}+\cdots+\frac{1}{x_n}} \geq n^{n(x_{n+1})^{\frac{1}{n}}}$ and (2) yields

$$S_{n+1} \geq n \cdot n^{x_{n+1}^{1/n}} = n \cdot n^{\sqrt[n]{x_{n+1}}}.$$

In the same way, if we set $S_j = S - n^{1/x_j}, \quad (j = 1, \ldots, n)$, we obtain

$$S_j \geq n \left(n^{n(x_j)^{\frac{1}{n}}}\right)^{\frac{1}{n}} = n \cdot n^{\sqrt[n]{x_j}}.$$

By addition,

$$S_1 + \cdots + S_{n+1} \geq n \left(n^{\sqrt[n]{x_1}} + \cdots + n^{\sqrt[n]{x_{n+1}}}\right)$$

and the result follows since $S_1 + \cdots + S_{n+1} = n(n^{1/x_1} + \cdots + n^{1/x_{n+1}})$.

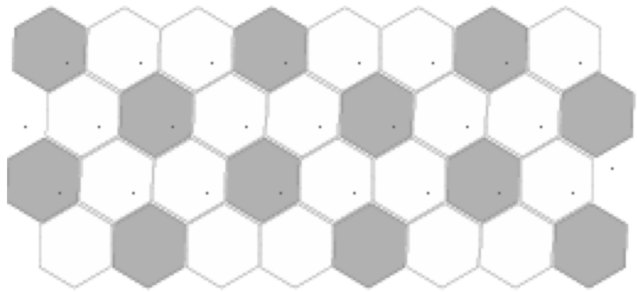
OC242. Let k be a positive integer. Two players A and B play a game on an infinite grid of regular hexagons. Initially all the grid cells are empty. Then the players alternately take turns with A moving first. In his move, A may choose two adjacent hexagons in the grid which are empty and place a counter in both of them. In his move, B may choose any counter on the board and remove it. If at any time there are k consecutive grid cells in a line all of which contain a counter, A wins. Find the minimum value of k for which A cannot win in a finite number of moves, or prove that no such minimum value exists.

Originally problem 5 from day 2 of the 2014 USAJMO.

We present the solution by Oliver Geupel. There were no other submissions.

The answer is $k = 6$.

First we show that A cannot win when $k = 6$. Colour the cells white and grey according to the following pattern, continued to infinity:



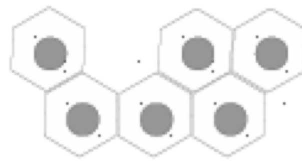
Player A can occupy only one grey cell per move. Player B's strategy is to remove the counter from the grey cell. Every arrangement of six cells in a line occupies at least two grey cells. By B's strategy, A cannot reach such a pattern. Hence A cannot win when $k = 6$.

In what follows we give a winning way for A when $k \leq 5$. Player A can reach an arrangement of 3 consecutive occupied cells in a line in two moves. In the next move A can create a pattern congruent to this one:

If B now would remove the counter from the lower cell, A could win immediately. Thus, B must remove a counter from the upper row. Then, A can reach an arrangement congruent to the following pattern P:

If B now would remove a counter from the lower row, A could win immediately. By reasons of symmetry, it is therefore enough to assume that B removes the upper-left or the upper-central counter. B should not remove the upper-left counter, since A could reach the pattern

which lets him win in the next move. Therefore, B must remove the upper-central counter in pattern P. Then, A moves to the following pattern Q:



To avoid immediate loss, B must remove a counter from the lower row in Q. If B removes the lower-left or the lower-central counter, A can reach to

which wins in the next move. Otherwise, B removes the lower-right counter in Q. Player A moves to

which ensures that A will win in the next move. We have proved that A has a winning strategy when $k \leq 5$.

OC243. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

Originally problem 4 of the 2014 France Team Selection Test.

We received 4 correct submissions and 1 incorrect submission. We present the solution by David Manes.

Clearly, $f(x) = x$ for all $x \in \mathbb{Z}_{>0}$ satisfies the property since $m = n = x$ implies $m^2 + f(n) = mf(m) + n = x^2 + x$.

Assume the condition is satisfied and let $m = n = 2$. Then $4 + f(2) \mid 2f(2) + 2$ implies $(4+f(2))k = 2f(2)+2$ for some positive integer k. Therefore, $f(2) = \frac{4k-2}{2-k}$. Note that $k \neq 2$ since $f(2) \in \mathbb{Z}_{>0}$. Also, if $k > 2$, then $f(2) < 0$, a contradiction. Hence, $k = 1$ so that $f(2) = 2$. Let $m = 2$ and $n = 1$. Then $4 + f(1) \mid 5$ implies $f(1) = \frac{5-4k}{k}$ for some positive integer k. Again, the only value of k for which $f(k)$ is a positive integer is $k = 1$, whence $f(1) = 1$. Assume inductively that $v \geq 2$ is an integer and $f(v) = v$. Let $m = v$ and $n = v+1$. Then $v^2 + f(v+1) \mid v^2 + v + 1$ implies $(v^2 + f(v+1))k = v^2 + v + 1$ for some positive integer k. Solving for

$f(v+1)$, one obtains

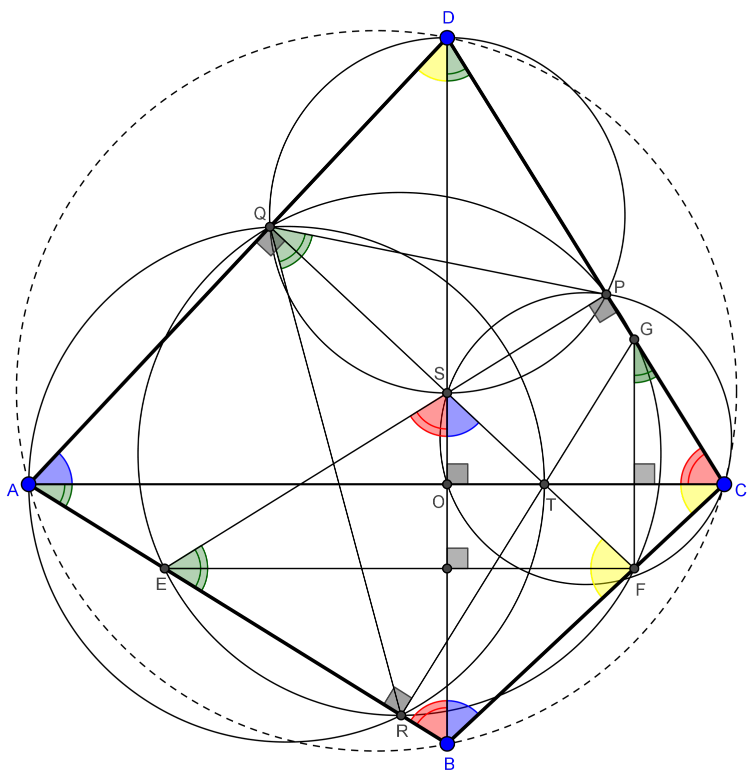
$$f(v+1) = \frac{v^2(1-k)+v+1}{k}$$

If $k > 1$, then $f(v+1) < 0$ for $v \geq 2$, a contradiction. Hence, $k = 1$ so that $f(v+1) = v+1$. Therefore, $f(x) = x$ for all positive integers x by induction.

OC244. $ABCD$ is a cyclic quadrilateral, with diagonals AC, BD perpendicular to each other. Let point F be on side BC, the parallel line EF to AC intersect AB at point E, line FG parallel to BD intersect CD at G. Let the projection of E onto CD be P, projection of F onto DA be Q, projection of G onto AB be R. Prove that QF bisects $\angle PQR$.

Originally problem 1 from day 1 of the 2014 China Team Selection Test.

We received 4 correct submissions. We present the solution by Andrea Fanchini.



We first note that $PGRE$ and $PGFE$ are cyclic so $PGFRE$ is cyclic.

$ABCD$ is cyclic so $\angle BDC = \angle BAC$, then $AC \parallel EF$ implies that $\angle BAC = \angle BEF$ and $FG \parallel BD$ implies that $\angle FGC = \angle BDC$. Therefore, $\angle BEF = \angle FGC$.

$PGFE$ is cyclic so $\angle PEF + \angle PGF = 180^\circ$, but $\angle PGF = 180^\circ - \angle FGC$. Therefore, $\angle PEF = \angle FGC = \angle BEF$ and $PF = FR$.

Now, $SOCP$ is cyclic so $\angle BSE = \angle OCP$, but $ABCD$ is cyclic so

$$\angle ACD = \angle OCP = \angle ABD = \angle EBS \quad \Rightarrow \quad \angle BSE = \angle EBS.$$

Therefore $\triangle BES$ is isosceles and furthermore as seen $\angle PEF = \angle BEF$ so EF is the perpendicular bisector of BS and also $\triangle FBS$ is isosceles. Then

$$\angle SFE = \angle EFB = \angle ACB = \angle ADB.$$

Now, as shown in the picture, yellow and blue angles (single arch notation) are complementary and the sum of the red and green angles (double arch notation) is 90°. So we have

$$\underbrace{\angle QAE}_{blue+green} + \underbrace{\angle AEF}_{180^\circ - green} + \underbrace{\angle EFS}_{yellow} = 270^\circ$$

therefore $FS \perp AD$ which implies S lies on FQ.

Let T be the intersection of AC with GR, we have similarly that also T lies on FQ. Note that $RTQA$ and $PDQS$ are cyclic. Then $\angle RQT = \angle RAT = \angle BAC$ and $\angle PQS = \angle PDS = \angle BDC$ but $\angle BAC = \angle BDC$ so $\angle RQT = \angle PQS$ and we are done.

OC245. Find all sets of 2014 not necessarily distinct rationals such that if we remove an arbitrary number in the set, we can divide the remaining 2013 numbers into three sets such that each set has exactly 671 elements and the product of all elements in each set is the same.

Originally problem 3 from day 2 of the 2014 Vietnam National Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

A multiset M of 2014 rationals satisfies the conditions of the problem if and only if:

1. M contains at least four occurrences of 0, or
2. all elements of M have the same absolute value and either all elements are equal or there are at least three negative and at least three positive ones.

First we show that every M with 1. or 2. satisfies the requirement.

If M satisfies condition 1. then after removing any element we can form three 671-element submultisets such that each of them contains an occurrence of 0. Hence M satisfies the requirement.

Now assume that M satisfies condition 2. If all elements are equal, everything is fine. After removing any element, we have m occurrences of a negative number $-q$, and $2013 - m$ occurrences of q, where $2 \leq m \leq 2011$. If m is an even number, then there are integers a, b, c such that $0 \leq a, b, c \leq 335$ and $m = 2(a + b + c)$.

Put $2a$, $2b$, and $2c$ negative elements in the first, second, and third submultiset, respectively. If m is odd, then there are integers a, b, c such that $0 \leq a, b, c \leq 335$ and $m = 2(a + b + c) + 3$. Put $2a + 1$, $2b + 1$, and $2c + 1$ negative elements in the first, second, and third submultiset, respectively. So M satisfies the condition of the problem.

It remains to prove that every multiset with the required property satisfies 1. or 2. Suppose M satisfies the condition of the problem with p as the common product value of the three multisets.

First consider the case $0 \in M$. If all elements of M are 0, then 1. is satisfied. Now assume that M contains an element $x \neq 0$. After removing x we have $p = 0$. Hence M contains at least three occurrences of 0. On the other hand, after removing 0 we have again $p = 0$. Therefore, M contains at least four occurrences of 0, i.e., M satisfies 1.

It remains to consider the case $0 \notin M$.

For every rational number q, the set $M_0 = \{qx \mid x \in M\}$ also has the required property. Let us choose q such that $1 \in M_0$. Removing 1 from M_0, the product of the remaining elements is p^3 for some rational number p. Removing any other element x from M_0, the product of the remaining elements is r^3 for some rational number r. Thus, $p^3 = xr^3$, that is, $x = (p/r)^3$, i.e., every element of M_0 is the 3rd power of a rational. Observe that the multiset $M_1 = \{\sqrt[3]{x} \mid x \in M_0\}$ also has the required property. Repeating the observation, we obtain an infinite sequence of multisets $M_0,\ M_1,\ M_2,\ \ldots$ such that $M_k = \{\sqrt[3^k]{x} \mid x \in M_0\}$ and M_k has the required property. Since the elements are rationals, this is possible only if all elements of M_0 are ± 1.

If there were exactly one -1 or two 1's, there were no appropriate partition after removing a 1. If there were exactly two -1's or one 1, there were no appropriate partition after removing a -1. Hence condition 2. is satisfied.

Editor's Note. Congratulations to Oliver Geupel who managed to solve all 5 OC problems in this edition! Well done!