THE OLYMPIAD CORNER

No. 347

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **May 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC301. Solve the following Diophantine equation for integers x and y:

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

OC302. Let x, y and z be real numbers where the sum of any two among them is not 1. Show that,

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2} + \frac{(y^2+z)(y+z^2)}{(y+z-1)^2} + \frac{(z^2+x)(z+x^2)}{(z+x-1)^2} \geq 2(x+y+z) - \frac{3}{4}.$$

Find all triples (x, y, z) of real numbers satisfying the equality case.

OC303. Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. Point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

OC304. Let k be a fixed positive integer. Let $F(n)$ be the smallest positive integer bigger than kn such that $F(n) \cdot n$ is a perfect square. Prove that if $F(n) = F(m)$, then $m = n$.

OC305. Let p be a prime number for which $\frac{p-1}{2}$ is also prime, and let a, b, c be integers not divisible by p. Prove that there are at most $1 + \sqrt{2p}$ positive integers n such that $n < p$ and p divides $a^n + b^n + c^n$.

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OC301. Résoudre l'équation diophantienne

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3,$$

x et y étant des entiers.

OC302. Soit x, y et z des nombres réels dont les sommes deux à deux ne sont pas égales à 1. Démontrer que

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2}\geq 2(x+y+z)-\frac{3}{4}$$

et déterminer les triplets (x,y,z) qui vérifient l'égalité.

OC303. Soit ABC un triangle, H son orthocentre et O le centre du cercle circonscrit au triangle. Soit K le milieu de AH et P le point sur AC tel que $\angle BKP=90^\circ$. Démontrer que OP est parallèle à BC.

OC304. Soit k un entier fixe strictement positif. Soit $F(n)$ le plus petit entier strictement positif supérieur à kn tel que $F(n)\cdot n$ est un carré parfait. Démontrer que si $F(n)=F(m)$, alors $m=n$.

OC305. Soit p un nombre premier pour lequel $\frac{p-1}{2}$ est aussi un nombre premier et soit a,b et c des entiers qui ne sont pas divisibles par p. Démontrer qu'il existe au plus $1+\sqrt{2p}$ entiers strictement positifs n tels que $n<p$ et que p soit un diviseur de $a^n+b^n+c^n$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(7), p. 288–289.

OC241. Let n be a natural number. For every positive real numbers $x_1,x_2,...,x_{n+1}$ such that $x_1x_2...x_{n+1}=1$ prove that:

$$\sqrt[x_1]{n}+\cdots+\sqrt[x_{n+1}]{n}\geq n^{\sqrt[n]{x_1}}+\cdots+n^{\sqrt[n]{x_{n+1}}}.$$

Originally problem 5 from day 2 of the 2014 Iran Team Selection Test.

We received 2 correct submissions. We present the solution by Michel Bataille.

We shall use the following inequality of means: if $x,a_1,\ldots,a_n>0$, then

$$\left(\frac{a_1^x+\cdots+a_n^x}{n}\right)^{1/x}\geq\sqrt[n]{a_1\cdots a_n}$$

which rewrites as

$$a_1^x+\cdots+a_n^x\geq n(a_1\cdots a_n)^{x/n}. \qquad (1)$$