FOCUS ON...

No. 24

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Solutions to Exercises from Focus On... No. 17–21

From Focus On... No. 17

1. *Show that if k is a positive odd integer and $2^k + 3^k = a^n$ for some integers a, n with $n \geq 2$, then k is a multiple of 5.*

Since k is odd, $2^k + 3^k = (2+3) \cdot N = 5N$ where

$$N = 2^{k-1} - 3 \cdot 2^{k-2} + \cdots + 3^{k-1}.$$

It follows that 5 divides a^n, hence 5 divides a (since 5 is prime) and so 5^n divides $a^n = 2^k + 3^k = 5N$. Recalling that $n \geq 2$, we see that 5 divides N. However, since $2 \equiv -3 \pmod 5$ and k is odd, we have $2^{k-1} \equiv 3^{k-1} \pmod 5$ and more generally

$$(-1)^j 3^j \cdot 2^{k-j-1} \equiv 2^{k-1} \pmod 5.$$

Therefore $N \equiv k \cdot 2^{k-1} \pmod 5$ and so 5 divides $k \cdot 2^{k-1}$. Since 5 and 2^{k-1} are coprime, 5 divides k.

2. *Let m, n be integers such that $m > n \geq 1$ and suppose that $m(m+n) = k^2 + \ell^2$ and $n(m-n) = 2(k^2 - \ell^2)$ for some integers k, ℓ. Prove that m, n have the same parity.*

For the purpose of a contradiction, assume that m and n are of opposite parity. Then $m-n$ is odd and the second relation implies that n is even, say $n = 2r$. Thus, m is odd and the first relation then shows that k and ℓ are also of opposite parity, say $k = 2s$ and ℓ odd. We obtain $r(m-2r) = 4s^2 - \ell^2$, an odd integer, hence r must be odd and so $r+m$ is even, say $r+m = 2t$. Now, from $(2t-r)(2t+r) = 4s^2 + \ell^2$, we deduce $(2t)^2 - (2s)^2 = r^2 + \ell^2$, a contradiction since modulo 8, $r^2 + \ell^2 \equiv 2$ while $(2t)^2 - (2s)^2 \equiv 0$ or 4. The proof is similar if k is odd and ℓ is even.

3. *Find all odd positive integers a, b, c, d, n such that $a^2 + b^2 + c^2 + d^2 = 7 \cdot 4^n$.*

Suppose that (a, b, c, d, n) is a solution. The integers a^2, b^2, c^2, d^2 being congruent to 1 modulo 8, we have $a^2 = 1 + 8a_1, b^2 = 1 + 8b_1, c^2 = 1 + 8c_1, d^2 = 1 + 8d_1$ for some nonnegative integers a_1, b_1, c_1, d_1. Then $1 + 2(a_1 + b_1 + c_1 + d_1) = 7 \cdot 4^{n-1}$, which calls for $n = 1$ (since $7 \cdot 4^{n-1}$ must be odd). It follows that $a_1 + b_1 + c_1 + d_1 = 3$ and, up to permutations, $(a_1, b_1, c_1, d_1) = (0,0,0,3)$ or $(0,0,1,2)$ or $(0,1,1,1)$. This leads to $(a^2, b^2, c^2, d^2) = (1,1,1,25)$ or $(1,1,9,17)$ or $(1,9,9,9)$, up to permutations. Clearly, the second case cannot occur, and so $(a, b, c, d) = (1,1,1,5)$ or $(1,3,3,3)$ (up to permutations). Conversely, it is readily checked that taking $n = 1$, $(a, b, c, d) = (1,1,1,5)$ or $(1,3,3,3)$ or their permutations provides solutions.

From Focus On... No. 18

1. *Show that if p is an odd prime number, then $(p+1)(p+2)\cdots(2p-1) \equiv (p-1)! \pmod{p^2}$.*

Let $L = (p+1)(p+2)\cdots(2p-1)$. It is readily checked that

$$2L = (p-1)!\binom{2p}{p}.$$

But it is well-known that

$$\binom{2p}{p} = \binom{p}{0}^2 + \binom{p}{1}^2 + \cdots + \binom{p}{p-1}^2 + \binom{p}{p}^2$$

(this is true for any positive integer p, even if p is not prime). Since $\binom{p}{j}$ is a multiple of p when $j = 1, 2, \ldots, p-1$, we have $\binom{2p}{p} \equiv 2 \pmod{p^2}$. It follows that $2L \equiv 2(p-1)! \pmod{p^2}$, so $L \equiv (p-1)! \pmod{p^2}$ (p is odd, 2 and p are coprime).

2. *Let a, b be positive integers and p be any prime. Show that $a^p - b^p$ is either coprime to p or divisible by p^2.*

If p and $a^p - b^p$ are not coprime, then p divides $a^p - b^p$. From Fermat Little Theorem, we have $a^p \equiv a \pmod{p}$ and $b^p \equiv b \pmod{p}$, hence

$$a^p - b^p \equiv a - b \pmod{p}$$

and so p divides $a - b$. But $a^p - b^p = (a-b)N$ where

$$N = a^{p-1} + a^{p-2}b + \cdots + ab^{p-2} + b^{p-1} \equiv pb^{p-1} \equiv 0 \pmod{p}$$

(since $a \equiv b \pmod{p}$). Thus, p divides each of the integers $a - b$ and N, hence p^2 divides $(a-b)N = a^p - b^p$.

3. *Let m be a positive integer such that $p = 1 + 4m$ is a prime. Show that the square of $(2m)!$ is congruent to -1 modulo p.*

We remark that modulo $p = 4m + 1$, we have

$$2 \equiv -(4m-1),\ 3 \equiv -(4m-2), \ldots, 2m \equiv -(2m+1).$$

It follows that $(2m)! \equiv -(2m+1)(2m+2)\cdots(4m-1)$ from which we obtain

$$((2m)!)^2 \equiv -[2 \times 3 \times \cdots \times (2m)] \cdot [(2m+1)(2m+2)\cdots(4m-2)(4m-1)] = -(p-2)!$$

Now, Wilson's theorem gives

$$-1 \equiv (p-1)! = (p-1)(p-2)! \equiv -(p-2)!$$

and we may conclude $((2m)!)^2 \equiv -1 \pmod{p}$.

From Focus On... No. 20

1. *Prove the identity*

$$vw(v-w)+wu(w-u)+uv(u-v)+(v-w)(w-u)(u-v)=0$$

where u,v,w are complex numbers and deduce another proof of Hayashi's inequality.

The identity is readily checked and supposing that u,v,w are distinct, we have

$$\frac{vw}{(w-u)(u-v)}+\frac{wu}{(v-w)(u-v)}+\frac{uv}{(v-w)(w-u)}=-1.$$

Using the triangle inequality, we obtain

$$\frac{|v||w|}{|w-u||u-v|}+\frac{|w||u|}{|v-w||u-v|}+\frac{|u||v|}{|v-w||w-u|}\geq 1. \qquad (1)$$

Taking P as the origin and u,v,w as the respective complex affixes of A,B,C, we have $|u|=PA$, $|v|=PB$, $|w|=PC$ and $|u-v|=AB$, $|v-w|=BC$, $|w-u|=AC$ and (1) then yields Hayashi's inequality:

$$\frac{PA\cdot PB}{CA\cdot CB}+\frac{PB\cdot PC}{AB\cdot AC}+\frac{PC\cdot PA}{BC\cdot BA}\geq 1.$$

2. *Using complex numbers, prove the identity*

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=(a^2b+b^2c+c^2a-abc)^2+(ab^2+bc^2+ca^2-abc)^2$$

for real numbers a,b,c. Deduce that if a,b,c are the sidelengths of a triangle, then

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)>(a^3+b^3+c^3)^2.$$

Clearly, we have $(b^2+c^2)(c^2+a^2)(a^2+b^2)=|(b+ic)(c+ia)(a+ib)|^2$. Expanding the product $p=(b+ic)(c+ia)(a+ib)$ gives

$$p=abc-ab^2-bc^2-ca^2+i(a^2b+b^2c+c^2a-abc)$$

and the identity is obtained from $(b^2+c^2)(c^2+a^2)(a^2+b^2)=|p|^2$.

Using the well-known $2(X^2+Y^2)\geq(X+Y)^2$, this gives

$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)\geq(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2-2abc)^2$$

or

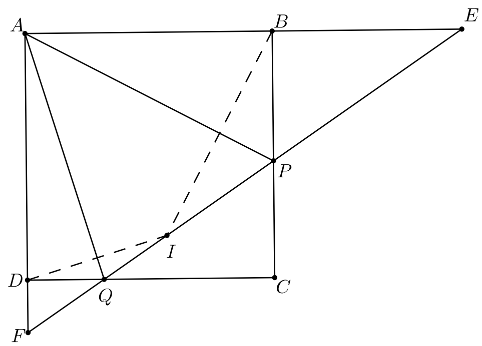
$$2(b^2+c^2)(c^2+a^2)(a^2+b^2)\geq(a^3+b^3+c^3+(a+b-c)(b+c-a)(c+a-b))^2.$$

Since a,b,c are the side lengths of a triangle, the product $(a+b-c)(b+c-a)(c+a-b)$ is positive and the desired inequality follows.

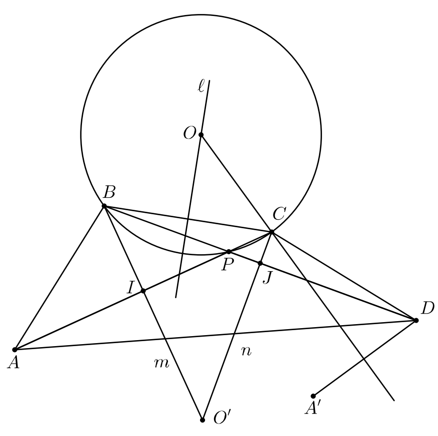
From Focus On... No. 21

1. *Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD, respectively, such that $\angle PAQ = 45°$. Let E and F be the intersections of PQ with AB and AD, respectively. Prove that $AE + AF \geq 2\sqrt{2}a$.*

The transformation $\mathbf{R}_{AQ} \circ \mathbf{R}_{AP}$ is the right angle rotation $\boldsymbol{\rho}_A$ with centre A such that $\boldsymbol{\rho}_A(B) = D$. Let $I = \boldsymbol{R}_{AP}(B)$. Since $AB \perp BP$, we have $AI \perp IP$. In addition, $\boldsymbol{R}_{AQ}(I) = \boldsymbol{R}_{AQ} \circ \boldsymbol{R}_{AP}(B) = \boldsymbol{\rho}_A(B) = D$, hence $I = \boldsymbol{R}_{AQ}(D)$ and so $AI \perp IQ$ as well. As a result, I is on the line PQ and I is the foot of the altitude from A in the right-angled triangle AEF. Since the hypotenuse EF is twice the median from A, we see that $EF \geq 2AI = 2a$ and so $AI \cdot EF \geq 2a^2$. Observing that $AI \cdot EF = AE \cdot AF$, the arithmetic-geometric mean inequality then gives $AE + AF \geq 2\sqrt{AE \cdot AF} = 2\sqrt{2}a$, as required.



2. *Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$ and such that AD and BC are not parallel. Let P be the intersection of the diagonals AC and BD. If $AP : BD = DP : AC$, prove that $AB \perp CD$.*



Following the given hints, we introduce the perpendicular bisectors ℓ, m, n of BC, CA, BD, respectively, and the circumcentre O of ΔBPC. Note that A

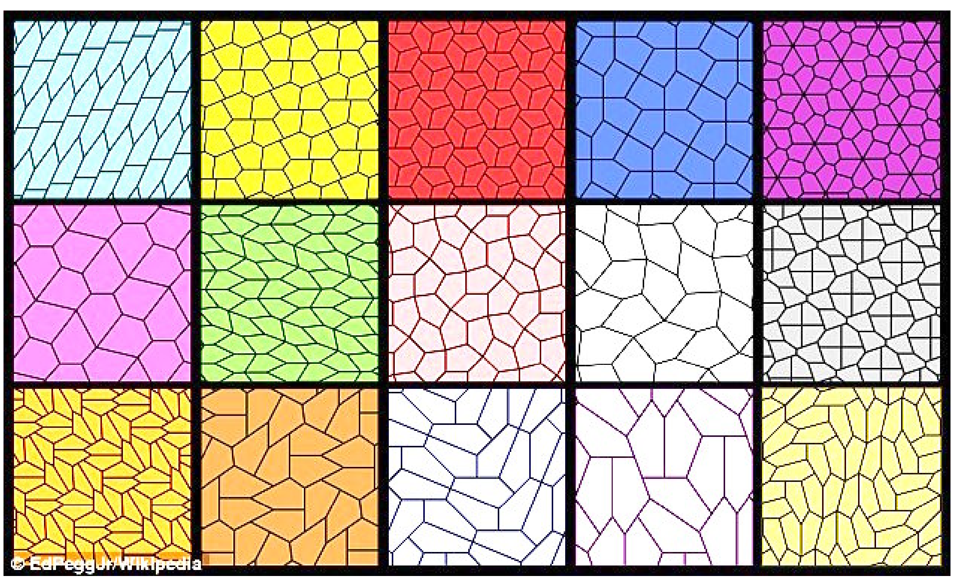
and D have the same power with respect to the circumcircle of ΔBPC (since $AP \cdot AC = DP \cdot DB$) and therefore $OA = OD$.

Since ℓ passes through O, $\boldsymbol{R}_{OC} \circ \boldsymbol{R}_\ell$ is a rotation $\boldsymbol{\rho}_O$ with centre O satisfying $\boldsymbol{\rho}_O(B) = \boldsymbol{R}_{OC}(C) = C$.

Let $A' = \boldsymbol{R}_\ell(A)$. Then $CA' = BA = CD$ and $OA' = OA = OD$ so that OC is the perpendicular bisector of DA'. It follows that $D = \boldsymbol{R}_{OC}(A') = \boldsymbol{\rho}_O(A)$. Since $\boldsymbol{\rho}_O(A) = D$ and $\boldsymbol{\rho}_O(B) = C$, the angle θ of $\boldsymbol{\rho}_O$ is $\theta = \angle(\overrightarrow{AB}, \overrightarrow{DC})$.

Similarly, since AC and BD are not parallel, m and n intersect and $\boldsymbol{R}_n \circ \boldsymbol{R}_m$ is a rotation $\boldsymbol{\rho}_{O'}$ with centre at the point O' of intersection of m and n. We have $\boldsymbol{\rho}_{O'}(A) = \boldsymbol{R}_n(C) = C$ and $\boldsymbol{\rho}_{O'}(B) = \boldsymbol{R}_n(B) = D$ so that the angle of $\boldsymbol{\rho}_{O'}$ is $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = \theta + \pi$.

Now, the midpoints I and J of AC and BD are on the circle with diameter $O'P$, hence $\angle(\overrightarrow{AB}, \overrightarrow{CD}) = 2\angle(m, n) = 2\angle(O'I, O'J) = 2\angle(PI, PJ) = 2\angle(PC, PB) = \angle(\overrightarrow{OC}, \overrightarrow{OB}) = -\theta$. Thus, $\theta + \pi = -\theta$ so that θ is a right angle and $AB \perp DC$.



There are now 15 known convex pentagons that can tile the plane. See problem CC241 in this issue to learn more about the latest discovery in the world of tiling.