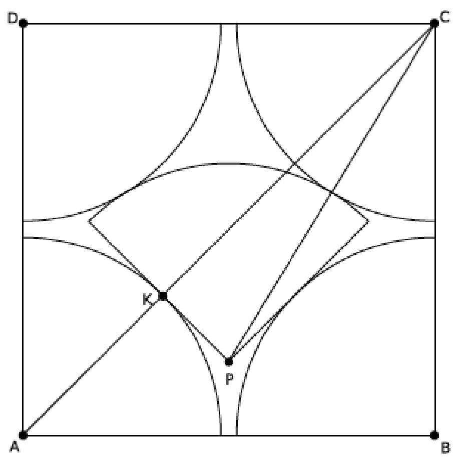
CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(7), p. 280–282.

CC181. A delivery boy decides to optimize the transportation of his pizzas. In his system, each box contains entirely and without any overlap 5 identical quarter pizzas (see below). The box is square, the drawing is symmetrical with respect to the center line and all contacts seen are mathematically perfect. The radius of a pizza is 16 cm. What is the minimal area of the bottom of the box rounded off to the nearest integer?

Originally problem 18 of the quarter-final of the 2012-13 Championnat International des Jeux Mathématiques et Logiques.

We received one correct solution by Ricard Peiró i Estruch which is presented below, modified by the editor.



Let $ABCD$ be the square forming the pizza box base with coordinates $A = (0, 0)$ and $C = (32+2x, 32+2x)$ where $2x$ is the distance between two of the corner pizza slices. Let K be the point where the bottom left pizza slice touches the central pizza slice and P be the point at the bottom of the central slice. Due to the symmetry of the slices and their right angle at the vertices, PK is perpendicular to AC and thus K lies on AC. We obtain $K = (8\sqrt{2}, 8\sqrt{2})$ and $P = (16 + x, 16\sqrt{2} - 16 - x)$. Finally $|PC| = 32$ and thus

$$32^2 = ((32 + 2x) - (16 + x))^2 + ((32 + 2x) - (16\sqrt{2} - 16 - x))^2.$$

We obtain

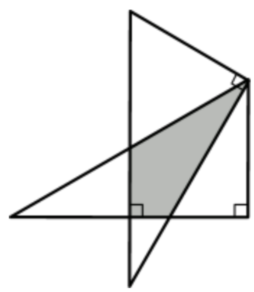
$$x = \frac{8\sqrt{38} + 24\sqrt{2} - 80}{5}$$

and

$$[ABCD] = (32 + 2x)^2 = \left(\frac{16\sqrt{38} + 48\sqrt{2}}{5}\right)^2 \approx 1109.$$

So the area of the pizza box is about 1109 cm^2.

CC182. We rearrange two halves of an equilateral triangle (cut along one of its altitudes) in the following way:

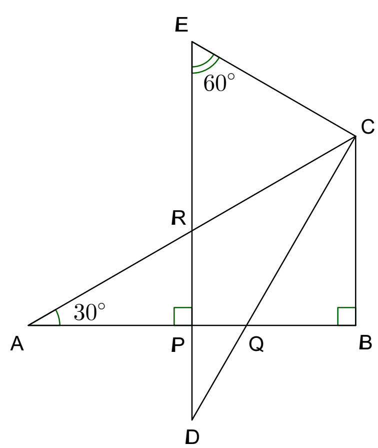


The area of the equilateral triangle was 600 cm^2. What is the area of the overlap region common to the 2 triangles?

Originally problem 14 of the 2012-13 quarter finals of the Championnat International des Jeux Mathématiques et Logiques.

We received five solutions. We present the solution by Ricard Peiró i Estruch.

Let $\triangle ABC$ and $\triangle DCE$ be our two right-angled triangles (with $\angle CAB = \angle EDC = 30^\circ$, and $\angle ABC = \angle DCE = 90^\circ$), as shown in the diagram below:



From the information given, the area of triangle $\triangle ABC$ (denoted $A_{\triangle ABC}$) is $\frac{1}{2} \cdot 600 = 300$. Denote the side BC by s; then, from standard equilateral triangle properties, $CE = s$, $AC = DE = 2s$, and $AB = DC = s\sqrt{3}$. Denote by PQCR the quadrilateral formed by the intersection of the two triangles (that is, the quadrilateral whose area we want to calculate).

Note that $\angle ARP = 60^\circ$, so $\angle ERC = 60^\circ$. It follows that $\triangle RCE$ is equilateral, so $RE = RC = s$, and also $AR = AC - RC = s$. Hence the similar triangles $\triangle APR$ and $\triangle ABC$ have side ratio $AR : AC = 1 : 2$. Therefore,

$$A_{\triangle APR} = \frac{1}{2^2} A_{\triangle ABC} = \frac{300}{4} = 75.$$

Since quadrilateral $PQCE$ is cyclic ($\angle EPQ = \angle ECQ = 90^\circ$), we have $\angle CQB = 60^\circ$. So $\triangle CQB$ is also similar to $\triangle ACB$ with side ratio $CB : AB = 1 : \sqrt{3}$. Therefore,

$$A_{\triangle CQB} = \frac{1}{\sqrt{3}^2} A_{\triangle ABC} = \frac{300}{3} = 100.$$

The area of quadrilateral $RPQC$ is the area of $\triangle ABC$ minus the areas of triangles $\triangle APR$ and $\triangle CQB$; that is,

$$A_{RPQC} = A_{\triangle ABC} - (A_{\triangle APR} + A_{\triangle CQB}) = 300 - (75 + 100) = 125 \text{ cm}^2.$$

CC183. We call a number *productive* if all the products of consecutive digits of the number can be found in its written form. 2013 and 1261 are examples of such numbers. Taking the first one as an example, we get the following consecutive products $2 \times 0 = 0, 0 \times 1 = 0$ and $1 \times 3 = 3$ which can all be found in the written form of 2013. For the second number, the products are $1 \times 2 = 2, 2 \times 6 = 12$ and $6 \times 1 = 6$ which can all be read in 1261. What is the smallest productive number which can be written using all the digits from 0 to 9?

Originally problem 17 of the semi-finals of the 2012-13 of the Championnat International des Jeux Mathmatiques et Logiques.

We received one correct submission. We present the solution by David Manes.

Starting with the numbers $10, 11, \ldots, 19, 20, 21, \ldots, 29, 30, 31$ and carefully trying to construct a productive number, but eliminating them for various and sundry reasons, we arrived at the number 3205486917, which easily satisfies the definition and is the smallest, hopefully, by construction. Some close, but no productive numbers were 2463180795 and 3154207698, the last digit in each case causing all the problems.

CC184. We are looking for two positive integers such that the difference of their squares is a cube and the difference of their cubes is a square. What is the value of the greatest of the two given that it is smaller than 20?

Originally problem 16 of the 2012-13 semi-finals of the Championnat International des Jeux Mathématiques et Logiques.

We received two solutions. We present the solution by David Manes.

Consider values of x and y, $1 \leq y < x \leq 19$, such that $x^2 - y^2 = n^3$ for some integer n. Then $3^2 - 1^2 = 2^3$, $6^2 - 3^2 = 3^3$, $10^2 - 6^2 = 4^3$, $14^2 - 13^2 = 3^3$, $15^2 - 3^2 = 6^3$, $15^2 - 10^2 = 5^3$ and $17^2 - 15^2 = 4^3$.

We now consider values of x and y, $1 \leq y < x \leq 19$ such that $x^3 - y^3 = n^2$ for some integer n. Then $8^3 - 7^3 = 13^2$, $10^3 - 6^3 = 28^2$ and $14^3 - 7^3 = 7^2$.

Accordingly, the only solution to the problem of finding two positive integers smaller than 20 such that the difference of their squares is a cube and the difference of their cubes is a square is 10 and 6 since $10^2 - 6^2 = 4^3$ and $10^3 - 6^3 = 28^2$. The value of the greatest is 10.

CC185. Each asterisk in the following multiplication can only be replaced by a digit in the set $\{2, 3, 5, 7\}$. Complete the multiplication.

$$\begin{array}{rrrrr}
 & & * & * & * \\
 & \times & & * & * \\
\hline
 & * & * & * & * \\
* & * & * & * & \\
\hline
* & * & * & * & *
\end{array}$$

Originally problem 16 of the 2011-2012 quarter final of the Championnat International des Jeux Mathématiques et Logiques.

We received three correct answers, one of which provided justification. We give the main steps.

The answer is $775 \times 33 = 25575$ with both partial products equal to 2325.

By checking what feeds into the last two digits of the product, we find that either the three-digit multiplicand or two-digit multiplier ends in 5. If the multiplicand ends in 5, each partial product ends in 5 and a check of the last two digits of the first partial product reveals that the only possibilities of the two factors are $(*24, *3)$, $(*75, *3)$, $(**5, *5)$, $(*25, *7)$. Checking various first digits for the multiplicand forces the partial products to be either $775 \times 3 = 2325$, $555 \times 5 = 2775$ or $325 \times 7 = 2275$.

If the multiplier ends in 5, a brief analysis shows that the multiplicand ends in 5. Thus, the possible factor pairs are $(775, 33)$, $(325, 77)$ and $(555, 55)$ and only the first of these works.