THE OLYMPIAD CORNER

No. 345

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The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by **April 1, 2017**.

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC291. Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum_{i=1}^n x_i = 1$. Prove that

$$\left(\sum_{i=1}^n \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

OC292. Consider two points with integer coordinates on the graph of a polynomial function with integer coefficients. If the distance between them is an integer, prove that the segment that connects them is parallel to the horizontal axis.

OC293. You are given $N \geq 3$. A set of N points on a plane is *acceptable* if their abscissae are unique, and each of the points is coloured either red or blue. A graph of a polynomial function $P(x)$ *divides* a set of acceptable points if there are no red dots above the graph of $P(x)$ and no blue dots below, or if there are no blue dots above the graph of $P(x)$ and no red dots below. Keep in mind, dots of both colours can be present on the graph of $P(x)$ itself. For what least value of k is an arbitrary acceptable set of N points divisible by a polynomial of degree k?

OC294. In given triangle $\triangle ABC$, the difference between sizes of each pair of sides is at least $d > 0$. Let G and I be the centroid and incenter of $\triangle ABC$ and r be its inradius. Show that

$$|AIG| + |BIG| + |CIG| \geq \frac{2}{3}dr,$$

where $|XYZ|$ is the area of triangle $\triangle XYZ$.

OC295. Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function that gives a positive integer value, to every positive integer. Suppose that f satisfies the following conditions:

$$f(1) = 1 \quad \text{and} \quad f(a + b + ab) = a + b + f(ab).$$

Find the value of $f(2015)$.

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OC291. Soit $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ des réels strictement positifs tels que $\sum_{i=1}^n x_i = 1$. Démontrer que

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1\leq i<j\leq n} x_i x_j\right) \leq \frac{n}{2}.$$

OC292. Sur la représentation graphique d'une fonction polynôme à coefficients entiers, deux points sont choisis avec des entiers pour coordonnées. Démontrer que si la distance entre les points est un entier, alors le segment qui les joint est parallèle à l'axe horizontal.

OC293. Soit N un entier $(n \geq 3)$. Un ensemble de N points dans le plan est appelé *acceptable* si les abscisses des points sont distinctes et si chacun des points est coloré en bleu ou en rouge. On dit qu'un ensemble acceptable de points dans le plan est *divisible* par la courbe représentative d'une fonction polynôme s'il n'y a aucun point rouge au-dessus de la courbe et aucun point bleu au-dessous de la courbe ou bien s'il n'y a aucun point bleu au-dessus de la courbe et aucun point rouge au-dessous de la courbe. À noter que des points de chaque couleur peuvent être situés sur la courbe. Quelle est la plus petite valeur de k pour laquelle n'importe quel ensemble acceptable de N points est divisible par un polynôme de degré k?

OC294. On considère un triangle ABC dont la différence entre les longueurs de chaque paire de côtés est supérieure ou égale à d $(d > 0)$. Soit G le centre de gravité du triangle, I le centre du cercle inscrit dans le triangle et r le rayon de ce cercle. Démontrer que

$$|AIG| + |BIG| + |CIG| \geq \frac{2}{3}dr,$$

$|XYZ|$ étant l'aire du triangle XYZ.

OC295. Soit $\mathbb{N} = \{1, 2, 3, \ldots\}$ l'ensemble des entiers strictement positifs et soit $f : \mathbb{N} \to \mathbb{N}$ une fonction à valeurs entières strictement positives qui satisfait aux conditions suivantes:

$$f(1) = 1 \quad \text{and} \quad f(a + b + ab) = a + b + f(ab).$$

Déterminer la valeur de $f(2015)$.

OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(5), p. 197–198.

OC231. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(x+y) + xy. \qquad (1)$$

for all $x, y \in \mathbb{R}$.

Originally problem 4 of the 2014 Balkan Mathematical Olympiad TST.

We received 8 correct submissions and 1 incorrect submission. We present the solution by Elnaz Hessami Pilehrood.

Substituting $x = 0$ in this equation, we get

$$f(0)f(y) = f(y) \qquad \text{or} \qquad f(y)(f(0) - 1) = 0.$$

Therefore, either $f(y) = 0$ or $f(0) = 1$.

We can see that $f = 0$ does not satisfy (1), so $f = 0$ cannot be such a function. Therefore, $f(0) = 1$.

When $f(0) = 1$, we can substitute $x = 1$ and $y = -1$ in (1) to get

$$f(1)f(-1) = f(0) - 1 = 0$$

and therefore, $f(1) = 0$ or $f(-1) = 0$.

If $f(1) = 0$, substitute $y = 1$ to get

$$f(x)f(1) = f(x+1) + x \qquad \text{or} \qquad 0 = f(x+1) + x,$$

which implies $f(x+1) = -x$ or $f(x) = 1 - x$. The function $f(x) = 1 - x$ satisfies all conditions, as $(1-x)(1-y) = 1 - x - y + xy$. If $f(-1) = 0$, substitute $y = -1$ to get

$$f(x)f(-1) = f(x-1) - x \qquad \text{or} \qquad f(x-1) = x.$$

The function $f(x) = x+1$ satisfies all conditions, as $(1+x)(1+y) = 1+x+y+xy$.

Therefore, all such functions are $f(x) = 1 - x$ and $f(x) = 1 + x$.

OC232. Given a positive integer m, Prove that there exists a positive integers n_0 such that all first digits after the decimal points of $\sqrt{n^2 + 817n + m}$ in decimal representation are equal, for all integers $n > n_0$.

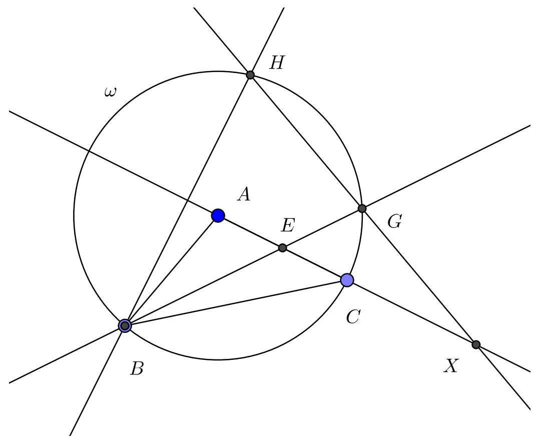
Originally problem 5 from day 2 of the 2014 China Western Mathematical Olympiad.

No submitted solutions.

OC233. Let ω be a circle with center A and radius R. On the circumference of ω four distinct points B, C, G, H are taken in that order in such a way that G lies on the extended B-median of the triangle ABC, and H lies on the extension of altitude of ABC from B. Let X be the intersection of the straight lines AC and GH. Show that the segment AX has length $2R$.

Originally problem 4 of the 2014 Italy Mathematical Olympiad.

We received 8 correct submissions. We present the solution by Somasundaram Muralidharan.



There is no loss of generality in assuming that the circle ω is $|z| = 1$ in the complex plane. Let the B, C, G, H be represented by the complex numbers b, c, g, h respectively. The slope of BG is given by $\dfrac{g-b}{\bar{g}-\bar{b}} = -bg$. The midpoint E of AC is $\dfrac{c}{2}$. Since G lies on the extension of BE, slope of BE must be equal to the slope of BG. Hence, we have

$$-bg = \frac{b - \frac{c}{2}}{\bar{b} - \frac{\bar{c}}{2}} = \frac{\frac{2b-c}{2}}{\frac{1}{b} - \frac{1}{2c}} = bc\left(\frac{2b-c}{2c-b}\right).$$

Thus,

$$g = -c\left(\frac{2b-c}{2c-b}\right).$$

Now the slope of AC is $\dfrac{c}{\bar{c}} = c^2$ and since BH is perpendicular to AC, slope of BH is $-c^2$. Since the slope of BH is also given by $\dfrac{b-h}{\bar{b}-\bar{h}} = -bh$, we obtain $-c^2 = -bh$ or $h = \dfrac{c^2}{b}$.

Now, the equation of AC is $Z = c^2\overline{Z}$ and that of HG is $Z - h = -hg(\overline{Z} - \overline{h})$. The lines HG and AC meet at X. Solving for $\overline{Z}$, we obtain $\overline{x}$, the conjugate of the complex number x representing X. Thus,

$$\overline{x} = \frac{h+g}{c^2+hg} = \frac{\frac{c^2}{b} - c\left(\frac{2b-c}{2c-b}\right)}{c^2 - \frac{c^3}{b}\left(\frac{2b-c}{2c-b}\right)} = \frac{2c^2 - bc - 2b^2 + bc}{bc(2c-b) - c^2(2b-c)} = \frac{2(c^2-b^2)}{c(c^2-b^2)} = 2\overline{c}.$$

Hence, $x = 2c$ and $AX = 2AC$. Thus, AX is twice the radius of ω.

OC234. Let N be an integer, $N > 2$. Arnold and Bernold play the following game: there are initially N tokens on a pile. Arnold plays first and removes k tokens from the pile, $1 \leq k < N$. Then Bernold removes m tokens from the pile, $1 \leq m \leq 2k$ and so on, that is, each player, on its turn, removes a number of tokens from the pile that is between 1 and twice the number of tokens his opponent took last. The player that removes the last token wins.

For each value of N, find which player has a winning strategy and describe it.

Originally problem 3 from day 1 of the 2014 Brazil National Olympiad.

No submitted solutions.

OC235. Prove that there is a constant $c > 0$ with the following property: If a, b, n are positive integers such that $\gcd(a+i, b+j) > 1$ for all $i, j \in \{0, 1, \ldots n\}$, then

$$\min\{a, b\} > c^n \cdot n^{\frac{n}{2}}.$$

Originally problem 6 from day 2 of the 2014 USA Mathematical Olympiad.

We present the solution by Oliver Geupel. There were no other submissions.

We prove the stronger bound

$$\min\{a, b\} > (cn)^n.$$

For $i, j \in \{0, 1, \ldots, n\}$, let p_{ij} be a prime such that $p_{ij} | \gcd(a+i, b+j)$. Then, for every prime number $p \in \{p_{ij} \mid i, j \in \{0, 1, \ldots, n\}\}$, the total number of pairs (i, j) such that $p = p_{ij}$, is not greater than $\left\lceil \frac{n+1}{p} \right\rceil^2 < \left(\frac{n+1}{p} + 1\right)^2$. Let n be a large integer and $N = (n+1)^2$. Then, the total number of pairs (i, j) such that $p_{ij} \leq N$, is not greater than

$$\sum_{p \leq N} \left(\frac{n+1}{p} + 1\right)^2 < (n+1)^2 \sum_{p \text{ prime}} \frac{1}{p^2} + 2(n+1) \sum_{p \leq N} \frac{1}{p} + \sum_{p \leq N} 1.$$

It is known that

$$\sum_{p \text{ prime}} \frac{1}{p^2} = 0.45\ldots < \frac{1}{2} \quad \text{and} \quad \sum_{p \leq N} \frac{1}{p} = O(\log \log N),$$

see [1] and [2], respectively. Also,

$$\sum_{p \leq N} 1 = O\left(\frac{N}{\log N}\right)$$

by the prime number theorem. Therefore, for every sufficiently large n, the total number of pairs (i, j) such that $p_{ij} \leq N$, is less than $(n+1)^2/2$.

By the pigeonhole principle there is an index $i \in \{0, 1, \ldots, n\}$ such that for more than half of the numbers $j \in \{0, 1, \ldots, n\}$ the respective prime p_{ij} is greater than N. Let i_0 denote such an index i. The primes $p_{i_0 j}$ that are greater than N are distinct and, therefore, are coprime divisors of $a + i_0$. A similar argument holds for the number b. We deduce that $\min\{a, b\} \geq N^{(n+1)/2} - n > n^n$ for every n exceeding some bound n_0. Let $c = 1/n_0$. For $n < n_0$ we have $\min\{a, b\} \geq 1 > (cn)^n$. For $n \geq n_0$ we obtain $\min\{a, b\} > n^n \geq (cn)^n$. Hence the result.

References

[1] The on-line encyclopedia of integer sequences, published electronically at https://oeis.org, sequence A085548.

[2] Tom M. Apostol, Introduction to analytic number theory, Springer 2010, theorem 4.12, page 90.