OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(2), p. 55–56.

OC216. Let $p = n^2 + 1$ be a given prime number. Find the set of integer solutions to the equation below:

$$x^2 - (n^2 + 1)y^2 = n^2.$$

Originally problem 4 from the number theory portion of the third round of the 2013 Iranian National Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Oliver Geupel.

We will show that the solutions for $p = 2$ are $x + y\sqrt{2} = \pm(3 + 2\sqrt{2})^k$, where $k \in \mathbb{Z}$. We will prove that the solutions for $p > 2$ are

$$\begin{aligned} x + y\sqrt{p} \in \{ &\pm (2n^2 + 1 + 2n\sqrt{p})^k n, \\ &\pm (2n^2 + 1 + 2n\sqrt{p})^k (n^2 - n + 1 + (n-1)\sqrt{p}), \\ &\pm (2n^2 + 1 + 2n\sqrt{p})^k (n^2 + n + 1 + (n+1)\sqrt{p}) \mid k \in \mathbb{Z}\}. \end{aligned}$$

For $z = x + y\sqrt{p} \in \mathbb{Z}[\sqrt{p}]$, denote $N(z) = x^2 - py^2$. We recapitulate the following well-known facts on Pell-like equations (check, e.g., the article *Pell's Equations* by Dušan Dukić on the website of the IMO compendium, retrieved March 25, 2016 from `http://imomath.com/index.php?options=615`):

(1) If z_1 is the fundamental solution (which must exist) of the equation $N(z) = 1$, i.e., the minimal element of $\mathbb{Z}[\sqrt{p}]$ with $z > 1$ and $N(z) = 1$, then all the solutions $z \in \mathbb{Z}[\sqrt{p}]$ are given by $z = \pm z_1^k$, $k \in \mathbb{Z}$.

(2) The fundamental solution of the equation $x^2 - 2y^2 = 1$ is $3 + 2\sqrt{2}$.

(3) For $a \in \mathbb{Z}$, every solution of the equation $N(z) = a$ has the form $z = \pm z_1^k z_a$ ($k \in \mathbb{Z}$) where z_1 is the fundamental solution of the equation $N(z) = 1$, and $z_a = x_a + y_a\sqrt{p}$ is a solution of $N(z) = a$ with $1 \le z_a \le z_1$. Also

$$|x_a| \le \frac{z_1 + 1}{2\sqrt{z_1}}\sqrt{|a|}.$$

The result for $p = 2$ follows from (1) and (2). It remains to consider $p > 2$.

Let $z_1 = x_1 + y_1\sqrt{p}$ be the fundamental solution of the equation $N(z) = 1$. We obtain $y_1^2 = (x_1 - 1)(x_1 + 1)/p$, that is, p is a divisor of either $x_1 - 1$ or $x_1 + 1$. We check the small values for x_1 in succession. If $x_1 = p - 1$ then $y_1^2 + 1 = n^2$, which is impossible. If $x_1 = p + 1$ then $y_1^2 = n^2 + 3$ which is impossible for $n > 1$. Trying $x_1 = 2p - 1$, we obtain $z_1 = 2n^2 - 1 + 2n\sqrt{p}$.

To complete the work, by (3) it is enough to find the solutions $z_a = x_a + y_a\sqrt{p}$ of $N(z_a) = a$ with $a = n^2$ and

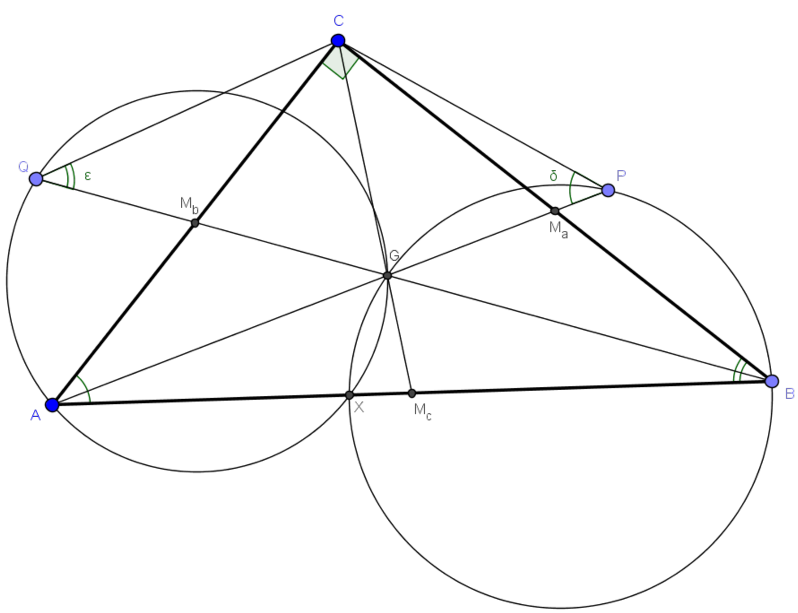
$$x_a < \frac{2n^2 + 2n\sqrt{n^2+1}}{2\sqrt{2n^2 - 1 + 2n\sqrt{n^2+1}}} \cdot n < 2n^2.$$

From $N(z_a) = a$ we have $y_a^2 = (x_a - n)(x_a + n)/p$, that is, p divides either $x_a - n$ or $x_a + n$. We check the small values for x_a in succession. Trying $x_a = n$, we find $z_a = n$. Putting $x_a = p - n$, we obtain $z_a = n^2 - n + 1 + (n-1)\sqrt{p}$. For $x_a = p + n$, we get $z_a = n^2 + n + 1 + (n+1)\sqrt{p}$. Trying $x_a = 2p - n$, we obtain $y_a^2 = 4(p - n)$. Hence, $p - n = n^2 + 1 - n$ is a perfect square, say m^2. We obtain $(n-1)^2 < m^2 < n^2$, a contradiction. The solution is complete.

OC217. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90°$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

Originally problem 3 of the 2013 Canadian Mathematical Olympiad.

We received 3 correct submissions. We present the solution by Andrea Fanchini.



We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC. With $\angle BCA = 90°$, we have $S_C = 0$ so $S_A = b^2$, $S_B = a^2$, $c^2 = a^2 + b^2$. Then a point P on ray AG has coordinates $P(u : 1 : 1)$ where u is a parameter. Now the oriented angle δ (with $0 \leq \delta \leq \pi$) between two lines

$d_i \equiv p_i x + q_i y + r_i z = 0 (i = 1, 2)$, is given from

$$S_\delta = S\cot\delta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

Therefore using the above, the angle between the line $CP \equiv x - uy = 0$ and the median $AP \equiv y - z = 0$, is

$$S_\delta = \frac{a^2 - 2b^2 u}{u + 2}.$$

But $\angle CPA = \angle CAB$, so

$$S_\delta = S_A = b^2 \quad \Rightarrow \quad u = \frac{a^2 - 2b^2}{3b^2} \quad \Rightarrow \quad P(a^2 - 2b^2 : 3b^2 : 3b^2).$$

Now the equation of a circle is $a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0$, so to find the equation of the circumcircle of $\triangle BPG$ we have to put in the coordinates of $B(0 : 1 : 0)$, $G(1 : 1 : 1)$, $P(a^2 - 2b^2 : 3b^2 : 3b^2)$ and solving the system, we have $p = b^2, q = 0, r = \frac{2a^2 - b^2}{3}$. Therefore, the intersection between the circumcircle BPG and the side AB gives the point X

$$\begin{cases} a^2yz + b^2zx + c^2xy - (x+y+z)(b^2x + \frac{2a^2-b^2}{3}z) = 0 \\ z = 0 \end{cases} \quad \Rightarrow \quad X(a^2 : b^2 : 0).$$

In the same way, we have $Q(1 : v : 1)$ and the angle between the median $BQ \equiv x - z = 0$ and the line $CQ \equiv vx - y = 0$, is

$$S_\epsilon = \frac{b^2 - 2a^2 v}{v + 2}.$$

But $\angle CQB = \angle ABC$, so

$$S_\epsilon = S_B = a^2 \quad \Rightarrow \quad v = \frac{b^2 - 2a^2}{3a^2} \quad \Rightarrow \quad Q(3a^2 : b^2 - 2a^2 : 3a^2).$$

To find the equation of the circumcircle of $\triangle AQG$, we have to put in the coordinates of $A(1 : 0 : 0)$, $G(1 : 1 : 1)$, $Q(3a^2 : b^2 - 2a^2 : 3a^2)$ and solving the system, we have $p = 0, q = a^2, r = \frac{2b^2 - a^2}{3}$. Therefore, the intersection between the circumcircle AQG and the side AB gives the point X'

$$\begin{cases} a^2yz + b^2zx + c^2xy - (x+y+z)(a^2y + \frac{2b^2-a^2}{3}z) = 0 \\ z = 0 \end{cases} \quad \Rightarrow \quad X'(a^2 : b^2 : 0).$$

So we have $X \equiv X'$, as required.

Note: this point is the intersection of the symmedian through vertex C and the side AB. The three symmedians concur at the Symmedian Point $K(a^2 : b^2 : c^2)$, well known also as Lemoine point.

OC218. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$

for all positive integers m, n.

Originally problem 5 from day 2 of the 2013 Korean National Olympiad.

We have a partial solution submitted by Konstantine Zelator which we include (and complete) here.

We claim that the only solutions are $f(m) = km$ for any $k \in \mathbb{N}$. First, this function is a solution since

$$\operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n)) = \frac{mn}{\gcd(m, n)} \cdot \gcd(km, kn) = kmn = f(mn).$$

Now, let's call the given relation $R_{m,n}$ and suppose $f(1) = c$. Then, $R_{m,1}$ for any $m \in \mathbb{N}$ gives

$$f(m) = m \gcd(f(m), c).$$

Now, by the relation $R_{cm,1}$ for any $m \in \mathbb{N}$ (and using the above), we see that

$$f(cm) = cm \cdot \gcd(f(cm), c) = cm \cdot \gcd(c \cdot \gcd(f(m), c), c) = c^2 m.$$

Next, the relation $R_{m,cn}$ for any $m, n \in \mathbb{N}$ reveals with the above two facts that

$$c^2 mn = f(mcn) = \operatorname{lcm}(m, cn) \cdot \gcd(f(m), f(cn)) = \frac{cmn}{\gcd(m, cn)} \cdot \gcd(f(m), c^2 n)$$

and simplifying yields

$$c \gcd(m, cn) = \gcd(f(m), c^2 n).$$

Therefore, $c \mid f(m)$. Hence, from the first derived relation above, we see that

$$f(m) = m \gcd(f(m), c) = mc$$

completing the proof.

OC219. Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

Originally problem 5 from day 2 of the 2013 USAMO.

We present the solution by Oliver Geupel. There were no other submissions.

Without loss of generality assume $m < n$. We consider the cases $\gcd(m, 10) = 1$ and $\gcd(m, 10) > 1$ in succession.

First, consider the case $\gcd(m, 10) = 1$. We have $\gcd\left(10^{n^2} n - m, 10\right) = 1$. Hence, there exists an integer $a > 3n^2$ such that $10^a \equiv 1 \left(\bmod\ 10^{n^2} n - m\right)$. Fix such an integer a and put $b = a - 2n^2$. As a consequence, we have $b > n^2$. Note that $m \equiv 10^{n^2} n \pmod{10^{n^2} n - m}$. Hence,

$$10^{n^2+b} m \equiv 10^{2n^2+b} n \equiv n \pmod{10^{n^2} n - m},$$

that is, the number $d = \dfrac{10^{n^2+b} m - n}{10^{n^2} n - m}$ is an integer.

We are going to show that

$$c = 10^{3b} + d$$

has the desired property.

By direct computation, we have

$$cm = 10^{3b} m + 10^{n^2} \cdot \frac{10^b m^2 - n^2}{10^{n^2} n - m} + n, \qquad cn = 10^{3b} n + 10^b m + \frac{10^b m^2 - n^2}{10^{n^2} n - m}.$$

Thus,

$$f = \frac{10^b m^2 - n^2}{10^{n^2} n - m}$$

is an integer. We have $0 < f < 10^b$. Consequently, the decimal expansion of cm is the concatenation of the decimal expansions of the three numbers m, f, and n, padded with some additional zeros. Also, the decimal representation of cn is the concatenation of the decimal representations of n, m, and f, padded with some extra zeros. This completes the proof in the case $\gcd(m, 10) = 1$.

It remains to examine the case $\gcd(m, 10) > 1$. Let $m = 2^h 5^j k$ with nonnegative integers h, j, and k where $\gcd(k, 10) = 1$. For positive integers x and y, we write $x \sim y$ if the decimal representations of x and y have the same number of occurrences of each nonzero digit. We know that there is a positive integer c_0 such that $c_0 k \sim c_0 2^j 5^h n$. We obtain

$$2^j 5^h c_0 m = c_0 10^{h+j} k \sim c_o k \sim 2^j 5^h c_0 n.$$

Consequently, the number

$$c = 2^j 5^h c_0$$

has the required property for m and n.

OC220. Let $A_1 A_2 \ldots A_8$ be a convex octagon such that all of its sides are equal and its opposite sides are parallel. For each $i = 1, \ldots, 8$, define B_i as the intersection between segments $A_i A_{i+4}$ and $A_{i-1} A_{i+1}$, where $A_{j+8} = A_j$ and $B_{j+8} = B_j$ for all j. Show some number i, amongst 1, 2, 3, and 4 satisfies

$$\frac{A_i A_{i+4}}{B_i B_{i+4}} \leq \frac{3}{2}.$$

Originally problem 6 of the 2013 Mexican National Olympiad.

No submitted solutions.