Where does the altitude touch down?

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When solving problems from stereometry, it is important to correctly determine the arrangement and positioning of the figures involved. Many mistakes made when solving these problems involve the altitude of a pyramid or a prism. If you incorrectly place the base of the altitude in relation to the base of the given pyramid or prism, as a result you will incorrectly construct the angle between the base and the lateral edge and incorrectly determine its measure, among other mistakes.

The goal of this article is to teach the reader to solve stereometry problems while avoiding aforementioned mistakes. As such, we begin the article by stating three problems with the list of potential solutions. Of course, you should not try to guess the correct answer! By analyzing these three problems, you will then easily understand the solutions of the examples that will follow.

Problem 1. Given a pyramid $SABC$, where on the plane containing the base ABC does the pyramid's altitude touch down if

a) lateral edges are congruent;

b) lateral edges form equal angles with the base of the pyramid;

c) lateral faces form equal angles with the base of the pyramid;

d) vertex angles at the apex S are all right angles;

e) two pairs of opposite edges are mutually perpendicular;

f) lateral edges are congruent and $\angle ASB = \angle ASC = 60°$, $\angle BSC = 90°$;

g) $\angle ACB = 90°$ and $AC \perp BS$;

h) $\overline{SO} = \frac{1}{3}(\overline{SA} + \overline{SB} + \overline{SC})$ where O is the center of the base?

Possible answers:

1. the point of intersection of the medians of ABC;
2. the point of intersection of the altitudes of ABC;
3. the center of the circumcircle of ABC;
4. the center of the incircle of ABC;
5. the center of the incircle of ABC or the center of one of the excircles of ABC;
6. the midpoint of BC;
7. a point on the line containing BC.

Often to correctly solve the problem, you have to determine whether the foot of the pyramid's altitude falls on the inside or the outside of the pyramid.

Problem 2. In a pyramid $SABC$, lateral faces form equal angles with the base of the pyramid. Which of the following conditions is sufficient to guarantee that the foot of the pyramid's altitude is inside the triangle base ABC?

Possible answers:

1. $\triangle ABC$ is isosceles;
2. $\triangle ABC$ is equilateral;
3. $\triangle ABC$ is a right-angle triangle or is an acute-angle triangle;
4. $\triangle ABC$ is an obtuse-angled triangle;
5. all of the above.

Problem 3. The base of the pyramid $SABCD$ is a trapezoid $ABCD$ with $BC \parallel AD$ and $|BC| < |AD|$. The lateral faces of the pyramid are congruent. Which of the following conditions is sufficient to guarantee that the foot of the pyramid's altitude is outside of the trapezoidal base $ABCD$?

Possible answers:

1. $\angle ABD \leq 90°$;
2. $\angle ABD > 90°$
3. the angle between $\overline{BD}$ and $\overline{CA}$ is greater than or equal to $90°$;
4. the angle between $\overline{BA}$ and $\overline{CD}$ is greater than $90°$;
5. none of the above.

Let us now consider two examples.

Example 1. The lateral edges of a triangular pyramid all have length l and two of the vertex angles at the apex of the pyramid are equal. Find the volume of the pyramid.

Solution. Let $SABC$ be the given pyramid with $\angle ASC = \angle BSC$ and $\angle ASB = \beta$. We immediately have that $\triangle ASC \cong \triangle BSC$ and hence $|AC| = |BC|$. Therefore, the foot O of the pyramid's altitude SO coincides with the center of the circumcircle of ABC (Problem 1a). Then we have three possibilities (see Problem 2):

1. $\angle ACB < 90°$ and the foot of the altitude is inside $\triangle ABC$ (Figure 1a);
2. $\angle ACB = 90°$ and the foot of the altitude is midpoint D of AB (Figure 1b);
3. $\angle ACB > 90°$ and the foot of the altitude is outside $\triangle ABC$ (Figure 1c).

Case a). We have:

$$|AB| = 2l\sin\frac{\beta}{2}, \quad |SD| = l\cos\frac{\beta}{2}, \quad |AC| = |BC| = 2l\sin\frac{\alpha}{2},$$

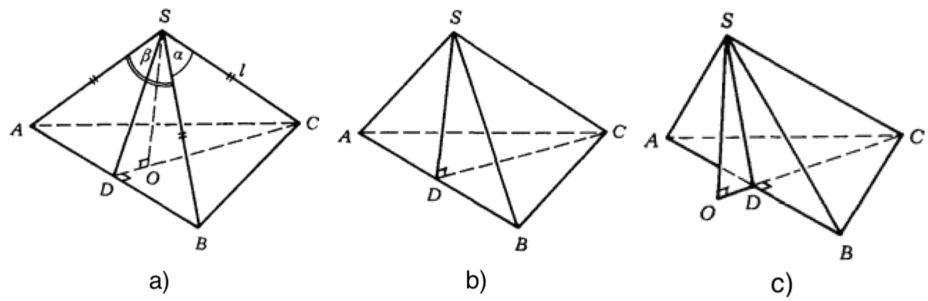

Figure 1: Possible positionings for the pyramid's altitude in Example 1.

$$|CD| = \sqrt{4l^2 \sin^2 \frac{\alpha}{2} - l^2 \sin^2 \frac{\beta}{2}} = l\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}.$$

Let $|OD| = x$. From the triangle SOD, we have

$$|SO|^2 = l^2 \cos^2 \frac{\beta}{2} - x^2.$$

From triangle CSO, we have

$$\begin{aligned} |SO|^2 &= l^2 - (|CD| - x)^2 \\ &= l^2 - l^2 + 2l^2 \cos\alpha - l^2 \cos^2 \frac{\beta}{2} - x^2 + 2xl\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}. \end{aligned}$$

Combining the above two equations and doing some manipulations, we get

$$x = l\frac{\cos^2 \frac{\beta}{2} - \cos\alpha}{\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}},$$
$$h = l\frac{\sqrt{\cos^2 \frac{\beta}{2} - \cos^2 \alpha}}{\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}}.$$

Therefore,

$$V = \frac{1}{3}\mathcal{S}_{\triangle ABC} \cdot h = \frac{1}{3}l^3 \sin\frac{\beta}{2}\sqrt{\cos^2 \frac{\beta}{2} - \cos^2 \alpha}. \tag{1}$$

Note that since $\angle ASB < \angle ASC + \angle BSC$, we have that $\beta < 2\alpha$ and hence the expression under the square root is positive.

Case b). This case is possible if and only if

$$\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}. \tag{2}$$

Indeed, if $\angle C = 90°$, we have $|CD| = |AD| = l\sin\frac{\beta}{2}$ and $|CD| = |AC|\frac{\sqrt{2}}{2} = \sqrt{2}l\sin\frac{\alpha}{2}$ (see Figure 1b). Then $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$. Conversely, if $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$, then

$$|CD| = l\sqrt{1 - 2\cos\alpha + \cos^2\frac{\alpha}{2}} = \sqrt{2}l\sqrt{1 - \cos\alpha - \sin^2\frac{\alpha}{2}} = \sqrt{2}l\sin\frac{\alpha}{2}.$$

Therefore, $\cos\angle ACD = \frac{|CD|}{|AC|} = \frac{\sqrt{2}}{2}$. Hence, $\angle ACD = 45°$ and $\angle C = 90°$. If $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$, then $h = |SD| = l\cos\frac{\beta}{2}$ (Figure 1b) and

$$V = \frac{l^2}{3}\sin^2\frac{\beta}{2} \times l\cos\frac{\beta}{2} = \frac{l^3}{6}\sin\beta\sin\frac{\beta}{2}.$$

Case c). Let $x = |OD|$ (Figure 1c). We then find the values for x, h and V as in Case a).

We can see that the answers in Cases a) and c) are the same even though they arose from a different situation. Moreover, these answers also fit Case b), which is not hard to see: using (2), eliminate α in (1) to get (3). Therefore, in all cases $V = \frac{1}{3}l^3\sin\frac{\beta}{2}\sqrt{\cos^2\frac{\beta}{2} - \cos^2\alpha}$. (Let us emphasize, however, that the solution which *only* considers Case a) is not complete.) □

Example 2. Suppose the base of the pyramid $SABC$ is a triangle ABC with $|AB| = |BC| = 20$ and $|AC| = 32$. Lateral faces form equal angles of $45°$ with the base of the pyramid. Find the volume of the pyramid.

Solution. The foot of the pyramid's altitude is the point equidistant from the lines AB, AC and BC, that is, the center of the incircle of ABC or the center of one of the excircles of ABC (Problem 1c). Therefore, there are four pyramids which satisfy the conditions of the problem with points O_1, O_2, O_3 and O_4 as feet of the pyramid's altitudes (see Figure 2).

The area of the base is known to be $\mathcal{S}_{\triangle ABC} = 192$, so it remains to determine the lengths of the altitudes SO_1, SO_2, SO_3 and SO_4.

Case 1. Consider the pyramid with altitude SO_1. We have

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(20 + 20 + 32)r_1 = 36r_1 \iff 192 = 36r_1 \implies r_1 = h_1 = 5\frac{1}{3}.$$

Hence, $V_1 = 341\frac{1}{3}$.

Case 2. Pyramids with altitudes SO_2 and SO_3 are congruent. We have:

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(|AC|r_2 - |BC|r_2) = 16r_2 \iff 192 = 16r_2 \implies r_2 = h_2 = 12.$$

Hence, $V_2 = V_3 = 768$.

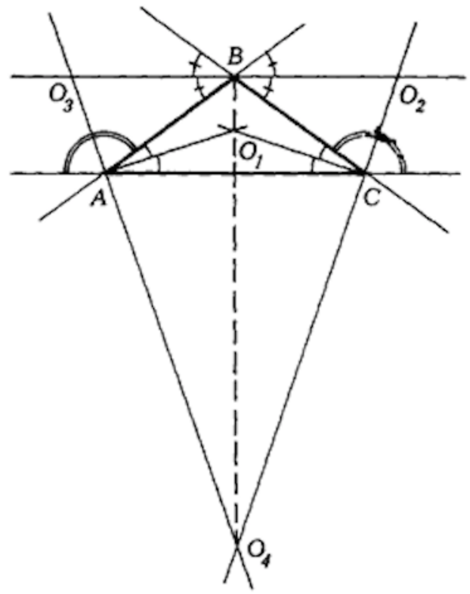

Figure 2: Possible positionings for the pyramid's altitude in Example 2.

Case 3. Finally, consider the pyramid with altitude SO_4. We have

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(|AB|r_4 + |BC|r_4 - |AC|r_4) = 4r_4 \iff 192 = 4r_4 \implies r_4 = h_4 = 48.$$

Hence, $V_4 = 3072$.

Therefore, the set of solutions is $\{341\frac{1}{3}, 768, 3072\}$.

Exercises.

1. The base of a pyramid is an isosceles trapezoid whose parallel sides are equal to a and $2a$. Lateral faces form equal angles with the base of the pyramid and pyramid's altitude equals a. Find the lateral surface area of the pyramid.

2. The base of a pyramid $SABC$ is a triangle ABC with $|AB| = a$ and $|BC| = b$. The lateral face which goes through the edge AC is perpendicular to the base of the pyramid. The other two lateral faces form equal angles with the base of the pyramid. Find the ratio of the volumes of pyramids $SABC$ and $SOBC$, where O is the foot of the altitude of the pyramid itself.

3. The base of a pyramid $SABC$ is a right-angle triangle ABC with hypotenuse c and acute angle α. The lateral edge SC forms an angle β with the base of the pyramid. The vectors $\overline{AO}$, $\overline{BO}$ and $\overline{CO}$ (where O is the foot of the altitude of the pyramid itself) connected end to end form a triangle. Find the volume of the pyramid.

4. Consider a parallelepiped with square base and top of side length b and rhombuses for all lateral faces. One of the vertices of the top face is equidistant from all the vertices of the bottom face. Find the volume of the parallelepiped.

5. A triangular prism has an isosceles right angle triangle ABC as its base with the hypotenuse BC and $|AB| = |AC| = a$. Lateral edges AA_1, BB_1 and CC_1 form an angle of 60° with the base of the prism. The diagonal BC_1 of the lateral face CBB_1C_1 is perpendicular to the edge AC. Finally, the length of the diagonal BC_1 is equal to $a\sqrt{6}$. Find the volume of the prism.

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Math Quotes

The mathematician who pursues his studies without clear views of this matter, must often have the uncomfortable feeling that his paper and pencil surpass him in intelligence.

Ernst Mach in "The Economy of Science" in J. R. Newman (ed.) "The World of Mathematics", New York: Simon and Schuster, 1956.