SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(4), p. 169–172.

4031. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Prove that

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

where F_n represents the nth Fibonacci number ($F_0=0, F_1=1$ and $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 1$).

We received five correct solutions. We present two solutions.

Editor's comments. When $n=1$, the interpretation of the left side is not clear, while when $n=2$, we obtain equality. Therefore, we suppose that $n\geq 3$.

Solution 1, by Adnan Ali and the proposers (independently).

Observe that, for positive x,y,z,

$$\frac{2x^2+y^2+z^2}{x+z}\geq x+y$$

with equality if and only if $x=y=z$, since this inequality is equivalent to $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$. It follows that the left side of the inequality is greater than

$$(F_1^2+F_2^2)+(F_2^2+F_3^2)+\cdots+(F_{n-1}^2+F_n^2)+(F_n^2+F_1^2)=2\sum_{k=1}^{n}F_k^2.$$

Since $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$ (easily obtained by induction for $n\geq 1$), the result follows.

Solution 2, by Arkady Alt.

For positive x,y,z,

$$\frac{x^2}{y+z}\geq\frac{4x-y-z}{4}$$

with equality iff $2x=y+z$. This implies that

$$\begin{aligned}\frac{2F_i^4+F_j^4+F_k^4}{F_i^2+F_k^2}&>\frac{2(4F_i^2-F_i^2-F_k^2)+(4F_j^2-F_i^2-F_k^2)+(4F_k^2-F_i^2-F_k^2)}{4}\\&=F_i^2+F_j^2,\end{aligned}$$

for distinct i, j, k (since not all of F_i, F_j, F_k are equal to $F_i + F_k$). Thus the left side is greater than $2\sum_{k=1}^{n} F_k^2 = 2F_nF_{n+1}$.

4032. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Prove that in any triangle ABC with sides a, b and c, inradius r and exradii r_a, r_b, r_c, we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(r_a + r_b + r_c)}.$$

We received 13 correct solutions. We present two solutions.

Solution 1, by Titu Zvonaru.

Using Ravi's substitutions ($a = y + z$, $b = z + x$, $c = x + y$, with $x, y, z > 0$), we have

$$[ABC] = \sqrt{xyz(x + y + z)},\ r = \sqrt{\frac{xyz}{x + y + z}},\ r_a = \frac{\sqrt{xyz(x + y + z)}}{x},$$

so that

$$r(r_a + r_b + r_c) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = xy + yz + zx.$$

We have to prove that

$$\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)} \geq 2\sqrt{3(xy + yz + zx)}.$$

Using Minkowski's inequality and the inequality $(x + y + z)^2 \geq 3(xy + yz + zx)$, we obtain

$$\begin{aligned}
&\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)}\\
&= \sqrt{x^2 + (xy + yz + zx)} + \sqrt{y^2 + (xy + yz + zx)} + \sqrt{z^2 + (xy + yz + zx)}\\
&\geq \sqrt{(x + y + z)^2 + (3\sqrt{xy + yz + zx})^2}\\
&\geq \sqrt{3(xy + yz + zx) + 9(xy + yz + zx)}\\
&= 2\sqrt{3(xy + yz + zx)}.
\end{aligned}$$

Equality holds if and only if $x = y = z$; that is, if and only if triangle ABC is equilateral.

Solution 2, composite of similar solutions by Šefket Arslanagic and Kee-Wai Lau.

By the known equality

$$r_a + r_b + r_c = 4R + r,$$

the given inequality is equivalent to

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(4R + r)}.$$

By the AM-GM inequality,

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 3 \cdot \sqrt[3]{abc}.$$

It therefore suffices to show that

$$3 \cdot \sqrt[3]{abc} \geq 2 \cdot \sqrt{3r(4R + r)},$$

which is successively equivalent to

$$\begin{aligned} 3 \cdot \sqrt[3]{4Rrs} &\geq 2 \cdot \sqrt{3r(4R + r)} \\ 3^6 \cdot 16R^2r^2s^2 &\geq 2^6 \cdot 27r^3(4R + r)^3 \\ 27R^2s^2 &\geq 4r(4R + r)^3. \end{aligned}$$

By the inequalities $s^2 \geq 16Rr - 5r^2$ due to J.C. Gerretsen and $R \geq 2r$ due to L. Euler, we have

$$\begin{aligned} 27R^2s^2 - 4r(4R + r)^3 &\geq 27(16Rr - 5r^2)R^2 - 4r(4R + r)^3 \\ &= r(R - 2r)(176R^2 + 25Rr + 2r^2) \\ &\geq 0. \end{aligned}$$

This proves the inequality of the problem. Equality holds for the equilateral triangle.

4033. *Proposed by Salem Malikic.*

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be positive real numbers and $x_1, \ldots, x_n$ be real numbers such that $x_1 + \cdots + x_n = 1$ and $\alpha_i x_i + \beta_i \geq 0$ for all $i = 1, \ldots, n$. Find the maximum value of

$$\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n}.$$

We received eight correct solutions and one incorrect solution. We present a composite of three nearly identical solutions given independently by Adnan Ali, Joe Schlosberg, and Titu Zvonaru.

Let M denote the required maximum value. Applying the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} &\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n} \\ &\qquad = \sqrt{\alpha_1}\sqrt{x_1 + \frac{\beta_1}{\alpha_1}} + \sqrt{\alpha_2}\sqrt{x_2 + \frac{\beta_2}{\alpha_2}} + \cdots + \sqrt{\alpha_n}\sqrt{x_n + \frac{\beta_n}{\alpha_n}} \\ &\qquad \geq \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(x_1 + x_2 + \cdots + x_n + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})} \\ &\qquad = \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(1 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})}. \end{aligned}$$

The equality holds if and only if for some k we have

$$\frac{x_1 + \frac{\beta_1}{\alpha_1}}{\alpha_1} = \frac{x_2 + \frac{\beta_2}{\alpha_2}}{\alpha_2} = \cdots = \frac{x_n + \frac{\beta_n}{\alpha_n}}{\alpha_n} = k.$$

Since

$$k = \frac{\sum x_i + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i} = \frac{1 + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i},$$

we have

$$x_i = \alpha_i \cdot \frac{1 + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i} - \frac{\beta_i}{\alpha_i}, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{1}$$

Therefore,

$$M = \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(1 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})}$$

attained when x_i are as in (1).

4034. *Proposed by Michel Bataille.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{16^n} \sum_{k=0}^{2n} \frac{(-1)^k}{2n+1-k} \binom{2k}{k} \binom{4n-2k}{2n-k}.$$

We received three correct solutions. We present the solution by Ángel Plaza.

Let us denote

$$a_n = \sum_{k=0}^{n} \frac{(-1)^k}{n+1-k} \binom{2k}{k} \binom{2n-2k}{n-k},$$

so the proposed expression reads as

$$\sum_{n=0}^{\infty} a_{2n} \left(\frac{-1}{16}\right)^n.$$

We will use the snake oil method to find the generating function of the sequence with general term a_n. If $F(x)$ is its generating function then

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} a_n x^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{(-1)^k}{n+1-k}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n \\
&= \sum_{k\geq 0}(-1)^k\binom{2k}{k}\sum_{n\geq k}\frac{1}{n+1-k}\binom{2n-2k}{n-k}x^n \\
&= \sum_{k\geq 0}(-1)^k\binom{2k}{k}x^k\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n} \\
&= \sum_{k\geq 0}\binom{2k}{k}(-x)^k\frac{2}{1+\sqrt{1-4x}} \\
&= \frac{1}{\sqrt{1+4x}}\cdot\frac{2}{1+\sqrt{1-4x}},
\end{aligned}$$

where we used the generating functions of the Catalan number and of the central binomial coefficients (both with radius of convergence $|x| < \frac{1}{4}$) in the last two steps. Now

$$\sum_{n\geq 0} a_{2n}x^n = \frac{F(\sqrt{x})+F(-\sqrt{x})}{2},$$

so the proposed sum is equal to

$$\frac{1}{\sqrt{1+i}}\cdot\frac{1}{1+\sqrt{1-i}}+\frac{1}{\sqrt{1-i}}\cdot\frac{1}{1+\sqrt{1+i}},$$

which can be calculated to $\sqrt{2}-\sqrt{\sqrt{2}-1}$.

4035. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a and b be two real numbers such that $ab = 225$. Find all real solutions (in real 2×2 matrices) to the matrix equation

$$X^3 - 5X^2 + 6X = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}.$$

We received four submissions for this question, of which three were correct and complete. We present the solution by Michel Bataille.

We will show that the solutions are the three matrices

$$X_1 = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

Simple calculations show that these three matrices satisfy the given equation.

Let $A = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}$ and let I_2 be the 2×2 unit matrix. Since $\det(xI_2 - A) = x^2 - 30x$, the eigenvalues of A are 0 and 30.

Let X be a solution of the given equation and let λ be an eigenvalue of X. Then $\lambda^3 - 5\lambda^2 + 6\lambda$ is an eigenvalue of A, hence $\lambda^3 - 5\lambda^2 + 6\lambda \in \{0, 30\}$. Thus,

$$\lambda(\lambda - 2)(\lambda - 3) = 0 \quad \text{or} \quad (\lambda - 5)(\lambda^2 + 6) = 0$$

and the possible eigenvalues of X are $0, 2, 3, 5, i\sqrt{6}, -i\sqrt{6}$.

Noting that the characteristic polynomial $\chi(x)$ of the real matrix X is in $\mathbb{R}[x]$, if $i\sqrt{6}$ (resp. $-i\sqrt{6}$) is an eigenvalue of X, so is its complex conjugate and then X is similar to $\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix}$. But then $A = X^3 - 5X^2 + 6X$ is similar to

$$\begin{pmatrix} -6i\sqrt{6} & 0 \\ 0 & 6i\sqrt{6} \end{pmatrix} - 5\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix} + 6\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix} = \begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix},$$

a contradiction since 0 is an eigenvalue of A.

As a result, if λ_1, λ_2 are the eigenvalues of X (possibly $\lambda_1 = \lambda_2$), we have $\lambda_1, \lambda_2 \in \{0, 2, 3, 5\}$. Since the eigenvalues of X are real, X is similar to an upper triangular real matrix, say,

$$X = P\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

for some invertible real matrix P and some real number α. Then

$$A = X^3 - 5X^2 + 6X = P\begin{pmatrix} \lambda_1^3 - 5\lambda_1^2 + 6\lambda_1 & \beta \\ 0 & \lambda_2^3 - 5\lambda_2^2 + 6\lambda_2 \end{pmatrix} P^{-1}$$

where $\beta \in \mathbb{R}$. Taking traces, it follows that

$$30 = (\lambda_1^3 - 5\lambda_1^2 + 6\lambda_1) + (\lambda_2^3 - 5\lambda_2^2 + 6\lambda_2) \qquad (1).$$

After trying all possibilities for λ_1 and λ_2, we realize that $\lambda_1 \neq \lambda_2$ and $\{\lambda_1, \lambda_2\}$ is either $\{0, 5\}$ or $\{2, 5\}$ or $\{3, 5\}$. We consider the three cases in turn:

- if $\{\lambda_1, \lambda_2\} = \{0, 5\}$, then $\chi(x) = x^2 - 5x$. Note that

  $$x^3 - 5x^2 + 6x = x(x^2 - 5x) + 6x = x\chi(x) + 6x.$$

  By the Cayley-Hamilton theorem, $\chi(X) = 0$; replacing x by X in the above we get $A = X\chi(X) + 6X = 6X$, hence $X = \frac{1}{6}A = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}$.

- if $\{\lambda_1, \lambda_2\} = \{2, 5\}$, then $\chi(x) = (x - 2)(x - 5)$ and

  $$x^3 - 5x^2 + 6x = (x + 2)\chi(x) + 10x - 20.$$

  Therefore $A = 10X - 20I_2$ and

  $$X = \frac{1}{10}(A + 20I_2) = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

- if $\{\lambda_1, \lambda_2\} = \{3, 5\}$, then $\chi(x) = (x-3)(x-5)$ and

$$x^3 - 5x^2 + 6x = (x+3)\chi(x) + 15x - 45.$$

Hence $A = 15X - 45$, which gives us

$$X = \frac{1}{15}(A + 45I_2) = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}.$$

The proof is complete.

4036. *Proposed by Arkady Alt.*

Let a, b and c be non-negative real numbers. Prove that for any real $k \geq \frac{11}{24}$ we have:

$$k(ab + bc + ca)(a + b + c) - (a^2c + b^2a + c^2b) \leq \frac{(3k-1)(a+b+c)^3}{9}.$$

We received five submissions all of which are correct. We present the solution by the proposer, slightly modified by the editor.

Due to cyclic symmetry of the functions involved, we may assume that $c = \min\{a, b, c\}$.

Let $x = a - c$ and $y = b - c$. Then $x, y, c \geq 0, a = x + c,\ b = y + c$, and $a + b + c = x + y + 3c$.

The given inequality is equivalent to

$$(3k-1)(a+b+c)^3 - 9k(ab+bc+ca)(a+b+c) + 9\left(a^2c + b^2a + c^2b\right) \geq 0$$

or

$$(3k-1)(x+y+3c)^3 - 9k((x+c)(y+c) + c(x+y+2c))(x+y+3c) \\ + 9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \geq 0. \quad (1)$$

Let $F(x, y, c)$ denote the expression on the left hand side of (1), and set $p = x + y$ and $q = xy$.
Since $9k((x+c)(y+c)+c(x+y+2c))(x+y+3c) = 9k(3c^2+2pc+q)(p+3c)$ and

$$\begin{aligned} &9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \\ &= 9(cx^2 + cy^2 + 2cxy + 3c^2(x+y) + xy^2 + 3c^3) \\ &= 9(cp^2 + 3c^2p + 3c^3 + xy^2) \\ &= 9cp^2 + 27c^2p + 27c^3 + 9xy^2 \\ &= (p+3c)^3 - p^3 + 9xy^2, \end{aligned}$$

we have

$$\begin{aligned}F(x,y,c) &= (3k-1)(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) + (p+3c)^3 - p^3 + 9xy^2\\ &= 3k(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) - p^3 + 9xy^2\\ &= 3k(p^3+9cp^2+27c^2p+27c^3) - 9k(2cp^2+9c^2p+9c^3+pq+3cq)\\ &\qquad - p^3 + qxy^2\\ &= (3k-1)p^3 + 9ckp^2 - 27ckq - 9kpq + 9xy^2\\ &= (3k-1)(x+y)^3 + 9ck(x+y)^2 - 27ckxy - 9kxy(x+y) + 9xy^2\\ &= (3k-1)(x^3+y^3+3xy(x+y)) + 9ck(x^2+2xy+y^2) - 27ckxy\\ &\qquad - 9kxy(x+y) + 9xy^2\\ &= (3k-1)x^3 + 6xy^2 - 3x^2y + 9ck(x^2-xy+y^2) + (3k-1)y^3. \qquad (2)\end{aligned}$$

Clearly, $9ck(x^2-xy+y^2)+(3k-1)y^3 \geq 0$. Furthermore,

$$(3k-1)x^3 + 6xy^2 - 3x^2y = x((3k-1)x^2 - 3xy + 6y^2) \geq 0$$

since the discriminant of $(3k-1)x^2 - 3xy + 6y^2$ is

$$9y^2 - 24(3k-1)y^2 = 3(11-24k)y^2 \leq 0$$

and $3k-1 > 0$. Hence, from (2) we conclude that $F(x,y,c) \geq 0$ which by (1) completes the proof.

4037. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that the area of UVW is independent of the chosen point P on γ.

We received six correct and complete solutions. We present the solution by Oliver Geupel. Ricard Peiró and Prithwijt De submitted similar solutions.

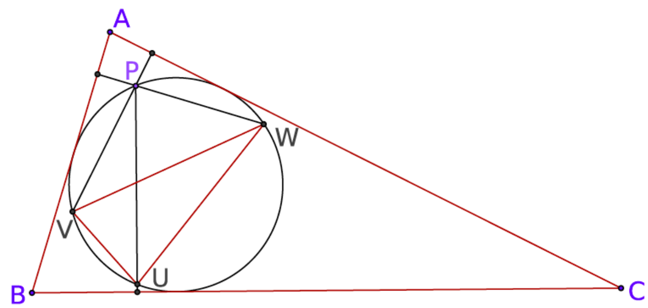
We prove that triangle UVW is similar to triangle ABC. As a consequence, since γ is the circumcircle of UVW, the area of triangle UVW is

$$[UVW] = \frac{r^2}{R^2}[ABC],$$

where r and R denote the inradius and the circumradius, respectively, of $\triangle ABC$.

Let $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ denote measures of the the interior angles of the triangles ABC and UVW. Since $PV \perp AC$ and $PW \perp AB$, the size of $\angle VPW$ is $180^\circ - \hat{A}$. Also, since the points P, U, V, and W are concyclic, $\angle VUW$ is equal to either $\angle VPW$ or $180^\circ - \angle VPW$. Hence, $\hat{U} \in \{\hat{A},\ 180^\circ - \hat{A}\}$. Analogously, $\hat{V} \in \{\hat{B},\ 180^\circ - \hat{B}\}$ and $\hat{W} \in \{\hat{C},\ 180^\circ - \hat{C}\}$.

We show that $(\hat{U}, \hat{V}, \hat{W}) = (\hat{A}, \hat{B}, \hat{C})$.



Assume the contrary. Then, there is no loss of generality in assuming that $\hat{U} \neq \hat{A}$. Thus, $\hat{U} = 180^\circ - \hat{A}$.

If $(\hat{V}, \hat{W}) = (\hat{B}, \hat{C})$, then we obtain $180^\circ = \hat{U} + \hat{V} + \hat{W} = 180^\circ - \hat{A} + \hat{B} + \hat{C}$, so that $\hat{A} = \hat{B} + \hat{C} = 90^\circ = 180^\circ - \hat{A} = \hat{U}$, contradicting our assumption that $\hat{U} \neq \hat{A}$. Hence, $(\hat{V}, \hat{W}) \neq (\hat{B}, \hat{C})$. There is no loss of generality in assuming that $\hat{V} = 180^\circ - \hat{B}$. But then, $\hat{U} + \hat{V} = (180^\circ - A) + (180^\circ - B) > 180^\circ$, a contradiction.

Editor's Comments. We were pleased to find that among the six solutions submitted, four different formulas for the area of a triangle were used.

4038. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find the minimum value of the expression

$$\sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1}.$$

There were 21 correct solutions, with four from one submitter and three from another. An additional solution was incorrect.

Solution 1, by Arkady Alt, Šefket Arslanagić, and Daniel Dan (independently).

Since $xyz = x+y+z \geq 3\sqrt[3]{xyz}$, if follows that $x+y+z = xyz \geq 3\sqrt{3}$. Applying the inequality of the root mean square and arithmetic mean, we have, for $t = x, y, z$,

$$\sqrt{\frac{1}{3}t^4 + 1} = \sqrt{\left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + 1}$$
$$\geq \frac{(t^2/3) + (t^2/3) + (t^2/3) + 1}{2} = \frac{t^2 + 1}{2}$$

with equality iff $t = \sqrt{3}$. (Alternatively, the inequality $\sqrt{\frac{1}{3}t^4 + 1} \geq \frac{t^2+1}{2}$ is equivalent to $(t^2 - 3)^2 \geq 0$.) Therefore, the left side of the inequality is not less than

$$\frac{1}{2}(3 + x^2 + y^2 + z^2) \geq \frac{1}{2}\left(3 + \frac{(x+y+z)^2}{3}\right) \geq \frac{1}{2}\left(3 + (27/3)\right) = 6.$$

Since equality occurs when $x = y = z = \sqrt{3}$, the desired minimum is 6.

Solution 2, by Šefket Arslanagić and Salem Malikić (independently).

As before, $x + y + z \geq 3\sqrt{3}$. We begin by establishing that

$$\sqrt{\frac{1}{3}u^4 + 1} \geq u\sqrt{3} - 1,$$

with equality iff $u = \sqrt{3}$. The result is clear when $u < 1/\sqrt{3}$. When $u \geq 1/\sqrt{3}$, the inequality is equivalent to

$$\frac{1}{3}u^4 + 2\sqrt{3}u \geq 3u^2.$$

By the arithmetic-geometric means inequality, we obtain that

$$\frac{1}{3}u^4 + 2\sqrt{3}u = \frac{1}{3}u^4 + u\sqrt{3} + u\sqrt{3} \geq 3\sqrt[3]{\frac{1}{3}u^4 \cdot u\sqrt{3} \cdot u\sqrt{3}} = 3u^2$$

as desired.

The left side of the inequality of the problem is not less than

$$\sqrt{3}(x + y + z) - 3 = 9 - 3 = 6,$$

with equality iff $x = y = z = \sqrt{3}$. The desired minimum is 6.

Solution 3, by Titu Zvonaru.

From the triangle inequality in Euclidean space $\mathbb{R}^2$,

$$\|\mathbf{a} + \mathbf{b} + \mathbf{c}\|_2 \leq \|\mathbf{a}\|_2 + \|\mathbf{b}\|_2 + \|\mathbf{c}\|_2,$$

applied to $\mathbf{a} = (x^2/\sqrt{3}, 1)$, $\mathbf{b} = (y^2/\sqrt{3}, 1)$, $\mathbf{c} = (z^2/\sqrt{3}, 1)$, we have that

$$\begin{aligned} \sqrt{\frac{x^4}{3} + 1} + \sqrt{\frac{y^4}{3} + 1} + \sqrt{\frac{z^4}{3} + 1} &\geq \sqrt{\left(\frac{x^2}{\sqrt{3}} + \frac{y^2}{\sqrt{3}} + \frac{z^2}{\sqrt{3}}\right)^2 + (1 + 1 + 1)^2} \\ &= \sqrt{\frac{(x^2 + y^2 + z^2)^2}{3} + 9}, \end{aligned}$$

with equality iff $x = y = z = \sqrt{3}$. Using $x^2 + y^2 + z^2 \geq xy + yz + zx$, the arithmetic-harmonic means inequality and $x + y + z = xyz$ in turn, we obtain that

$$x^2 + y^2 + z^2 \geq xy + yz + zx \geq \frac{9xyz}{x + y + z} = 9.$$

Thus the left side of the inequality is not less than

$$\sqrt{\frac{81}{3} + 9} = 6,$$

with equality iff $x = y = z = \sqrt{3}$. The desired minimum is 6.

Solution 4, by Ali Adnan.

By the Cauchy-Schwarz inequality,

$$\sqrt{\frac{1}{3}t^4+1}\cdot\sqrt{3+1}\ge t^2+1,$$

and by the arithmetic-geometric means inequality,

$$(x^2+y^2+z^2)(x+y+z)\ge 3(xyz)^{2/3}\cdot 3(xyz)^{1/3}=9xyz.$$

Therefore

$$\begin{aligned}\sqrt{\frac{1}{3}x^4+1}+\sqrt{\frac{1}{3}y^4+1}+\sqrt{\frac{1}{3}z^4+1} &\ge \frac{1}{2}(x^2+y^2+z^2)+\frac{3}{2}\\ &\ge \frac{1}{2}\left(\frac{9xyz}{x+y+z}\right)+\frac{3}{2}=6,\end{aligned}$$

with equality if and only if $x=y=z=\sqrt{3}$.

Editor's comments. Seven solvers applied Jensen's Inequality to obtain the result, the function $\sqrt{(x^4/3)+1}$ being convex. Some solvers noted that, under the stated constraint, we can write $(x,y,z)=(\tan\alpha,\tan\beta,\tan\gamma)$ with $\alpha+\beta+\gamma=\pi$ and each angle less than $\pi/2$, or $(x,y,z)=(\cot\lambda+\cot\mu+\cot\nu)$ with $\lambda+\mu+\nu=\pi/2$, and then obtain a trigonometric inequality.

4039. *Proposed by Abdilkadir Altinaş.*

In a triangle ABC, let $\angle CAB=48^\circ$ and $\angle CBA=12^\circ$. Suppose D is a point on AB such that $CD=1$ and $AB=\sqrt{3}$. Find $\angle DCB$.

We received 17 correct solutions and will feature the solution submitted by the Skidmore College Problem Group.

Let $\angle DCB=\gamma$; we shall show that $\gamma=6^\circ$.

By the Law of Sines applied to triangle ABC, $\frac{\sin 12^\circ}{AC}=\frac{\sin 120^\circ}{\sqrt{3}}=\frac{1}{2}$, so we have

$$AC=2\sin 12^\circ.$$

For triangle ACD, $\frac{\sin(12^\circ+\gamma)}{AC}=\frac{\sin 48^\circ}{1}$, whence

$$\sin(12^\circ+\gamma)=2\sin 48^\circ\sin 12^\circ=\cos 36^\circ-\cos 60^\circ=\cos 36^\circ-\frac{1}{2}.$$

Let $\phi=\frac{1+\sqrt{5}}{2}$ $(=\phi^2-1)$ be the golden section (that is, the ratio of a diagonal of the regular pentagon to a side). We know that

$$\sin 18^\circ=\frac{1}{2\phi}=\frac{\phi}{2}-\frac{1}{2}\quad\text{and}\quad\cos 36^\circ=\frac{\phi}{2}.$$

[For example, if the regular pentagon $PQRST$ has unit sides, then the isosceles triangle PRS has apex angle 36° and sides ϕ and 1 (which provides the value for

$\sin 18^\circ$), while the isosceles triangle PQT with base angle 36° also has sides 1 and ϕ (which gives $\cos 36^\circ$).] Thus $\cos 36^\circ = \sin 18^\circ + \frac{1}{2}$, and

$$\sin(12^\circ + \gamma) = \sin 18^\circ = \sin 162^\circ.$$

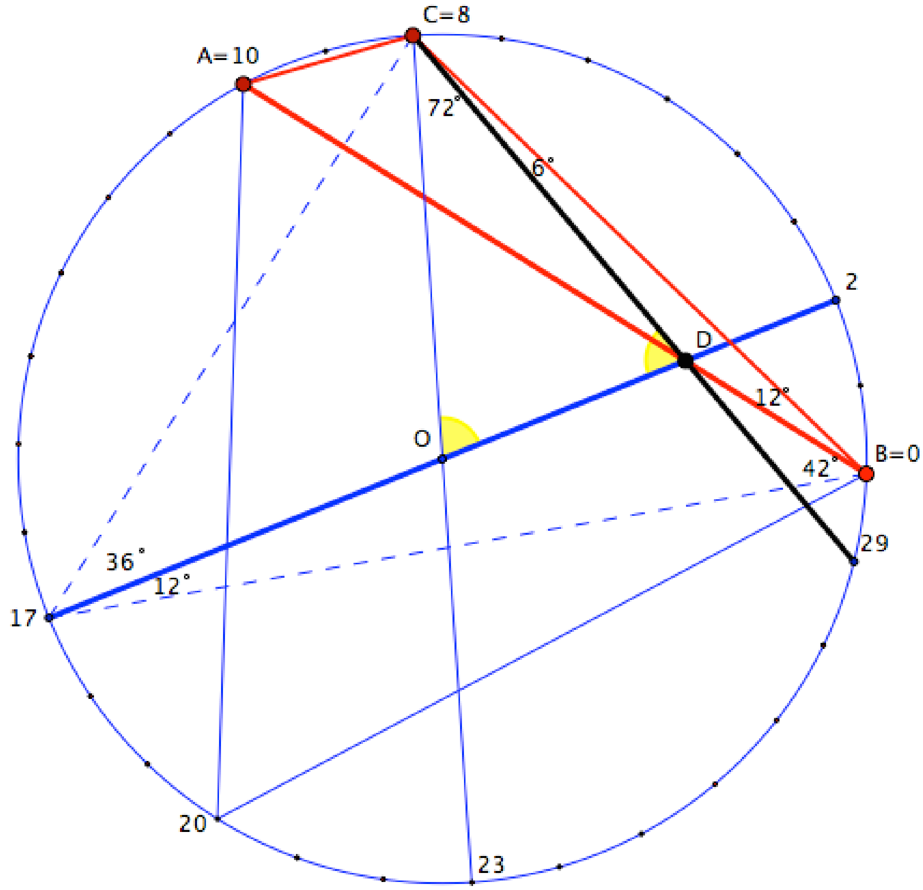
But $\gamma < 120^\circ$, which implies that $\gamma = 6^\circ$, as claimed.

Editor's comments. In a remark added to his solution, Dag Jonsson observed that there would be a second solution should we allow D to be a point of the line AB (instead of restricting it to the segment AB). Vertex A would then lie between D and B; the argument of the featured solution remains valid so that $\angle ADC = 18^\circ$, whence $\angle DCA = 30^\circ$ and, finally, $\angle DCB = 150^\circ$.

Returning to the case where D lies between A and B, we note that the degenerate quadrangle $ADBC$ of our problem is an example of an *adventitious quadrangle.* Adventitious quadrangles were first defined by Colin Tripp [5], but his narrow definition was later extended to include any quadrangle for which the angles formed by sides and diagonals are all rational multiples of 180°. Rigby [3] observed that the problem of classifying the adventitious quadrangles is equivalent to classifying all triple intersections of diagonals in regular polygons, a problem solved many years earlier by Gerrit Bol [1]. A complete account, including an elementary summary and a 15-item bibliography, was provided by Poonen and Rubinstein in [2]. Triangle ABC of our problem can be inscribed in a regular 30-gon that has a unit circumradius. Label its vertices from 0 to 29 and place A at 10, B at 0, and C at 8. Note that the angle subtended at a vertex of the 30-gon by any nonadjacent edge is 6°, which immediately implies that the angles of ΔABC are indeed $48^\circ, 12^\circ$, and 120°. Because AB is the edge of an inscribed equilateral triangle (with vertices numbered 0, 10, 20), we have $AB = \sqrt{3}$ as desired. Let D' be the point of intersection of the diameter 2, 17 and the diagonal 8, 29. If O is the circumcentre, then triangle $OD'C$ is isosceles (with angles $36^\circ, 72^\circ$, and 72°), whence $CD' = CO = 1$. It remains to prove that $D' \in AB$, which will immediately imply that $D' = D$ (and, consequently, that $\angle DCB = \angle D'CB = 6^\circ$). On p. 223 of [4] Rigby declares (with a slightly different labeling) that the diagonals 29, 8; 0, 10; and 2, 17 are indeed concurrent. While the author provides a "geometric" proof that these three lines are concurrent, it is perhaps more efficient to use trigonometry. Applying the sine form of Ceva's theorem to the triangle whose vertices are those numbered 0, 8, 17 with cevians joining 0 to 10, 8 to 29, and 17 to 2 (as in the accompanying figure), we must show that the product

$$\frac{\sin 42}{\sin 12} \cdot \frac{\sin 6}{\sin 72} \cdot \frac{\sin 36}{\sin 12}$$

equals 1. This is easily accomplished using an argument similar to that of our featured solution.



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4040. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$(f(a)+b)f(a+f(b)) = (a+f(b))^2 \quad \forall a, b \in \mathbb{N}.$$

There were nine submitted solutions for this problem, all of which were correct. We present three solutions; the first solution is for the case that 0 *is the first natural number, and the latter two are for the case that* 1 *is the first natural number.*

Solution 1, by Leonard Giugiuc.

Suppose that 0 is the first natural number. Let $a = b = 0$; we have:

$$f(0)f\big(f(0)\big) = f(0)^2,$$

which implies that either $f(0) = 0$ or $f\big(f(0)\big) = f(0)$. If $f\big(f(0)\big) = f(0)$, then we replace a with 0 and b with $f(0)$ in the given relation and obtain:

$$2f(0)f(0) = f(0)^2,$$

which implies that $f(0)^2 = 0$. Hence no matter what, $f(0) = 0$. Now replace b with 0 and obtain:

$$f(a)^2 = a^2,$$

for all $a \geq 0$, and so $f(a) = a$ for all $a \geq 0$, because $f(a) \geq 0$. Thus the required function is the identity (which clearly satisfies the given relation).

Solution 2, by Adnan Ali.

Suppose that 1 is the first natural number. Denote by $P(a,b)$ the above functional equation for all $a, b \in \mathbb{N}$. Now let $f(1) = k$, where k is a natural number. Then $P(1,1)$ gives $f(k+1) = k+1$. Using this, $P(1, k+1)$ implies

$$(2k+1)f(k+2) = (k+2)^2.$$

This means that $(2k+1)$ divides $(k+2)^2$. This means that $2k+1$ divides

$$4(k+2)^2 - 8(2k+1) - (2k+1)(2k-1) = 9.$$

This forces $2k+1$ to be either 3 or 9, which gives $f(1) = 1$ or $f(1) = 4$.

If $f(1) = 4$, then from $P(1,1)$ we get $f(5) = 5$. Next $P(5,1)$ gives

$$(f(5)+1)f(5+f(1)) = (5+f(1))^2 = 81,$$

implying that $6f(9) = 81$, which is clearly impossible as $6 \nmid 81$ and $f(9) \in \mathbb{N}$.

Thus we must have $f(1) = 1$. Now we prove by induction that $f(1) = 1$ implies that $f(n) = n$ for all $n \in \mathbb{N}$. The case $n = 1$ is already true. Now assume that $f(a) = a$ for some $a \geq 1$. Then from $P(a,1)$:

$$(f(a)+1)f(a+f(1)) = (a+f(1))^2,$$

which implies that $f(a+1) = a+1$, and consequently by induction $f(n) = n$ for all $n \in \mathbb{N}$. So, in summary the only function satisfying the equation is the function $f(n) = n$ for all $n \in \mathbb{N}$.

Solution 3, by Joseph Ling.

It is easy to see that $f(n) = n$ for all n satisfies the required relation for all $a, b \in \mathbb{N}$ We show that there are no other solutions.

First, we note that f is one-to-one. For if b_1 and b_2 are such that $f(b_1) = f(b_2)$, then for all a, we have

$$f(a) + b_1 = \frac{(a + f(b_1))^2}{f(a + f(b_1))} = \frac{(a + f(b_2))^2}{f(a + f(b_2))} = f(a) + b_2,$$

implying that $b_1 = b_2$.

Next, we note that f is (strictly) increasing. For if there exist $b_1 < b_2$ with $f(b_1) > f(b_2)$, then for every $a \in \mathbb{N}$, we consider $a' = a + f(b_1) - f(b_2)$. We have $a' \in \mathbb{N}$, with $a' + f(b_2) = a + f(b_1)$, and consequently,

$$f(a') + b_2 = \frac{(a' + f(b_2))^2}{f(a' + f(b_2))} = \frac{(a + f(b_1))^2}{f(a + f(b_1))} = f(a) + b_1 < f(a) + b_2$$

implying that $f(a') < f(a)$. But then this means that the range of f has no smallest element, contradicting the well-ordering principle.

Now, since f is strictly increasing, a simple induction shows that $f(n) \geq n$ for all $n \in \mathbb{N}$. It follows that $f(a + f(b)) \geq a + f(b)$ for all $a, b \in \mathbb{N}$. Consequently, the given relation implies that $f(a) + b \leq a + f(b)$ for all $a, b \in \mathbb{N}$. But then by symmetry, we conclude that in fact, $f(a) + b = a + f(b)$ for all $a, b \in \mathbb{N}$. Therefore,

$$f(n) = n + k$$

for all $n \in \mathbb{N}$, where $k = f(1) - 1$. Applying this to the relation, we get

$$(a + b + k)(a + b + 2k) = (a + b + k)^2$$

for all $a, b \in \mathbb{N}$. But then this implies that $k = 0$, and so, $f(n) = n$ for all $n \in \mathbb{N}$.

Editor's Comments. Most of the solvers, and the proposer, assumed simply that 0 was not a natural number; only Giugiuc solved the problem with both interpretations. Assuming that 0 is natural means there is more known information, and thus it is reasonable that the solution is easier than in the other case. Regarding the other case, the solution by Ali uses divisibility, a number-theoretic property, whereas the solution by Ling uses the function-theoretic properties of solutions to this relation. Other solutions use the relation differently (some use it to find additivity of the function, others show that there are infinitely many primes satisfying $f(n) = n$, etc.).