# Applications of Bertrand's postulate and its extensions in Math Olympiad style problems

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## 1 Introduction

Prime numbers are one of the fundamental entities in Number Theory. The guarantee of the existence of a prime number within a certain interval can be helpful in solving several types of problems. The Bertrand-Chebyshev theorem, also known as Bertrand's postulate, can be very useful in this context. Here we present several solved examples where it can be successfully used. A particular emphasis is placed on Math olympiad-style problems and therefore the article is based solely on elementary techniques. In addition to solved examples, this work also contains a brief historical and theoretical background, a list of some stronger results as well as a set of problems for self-study.

## 2 Historical and Theoretical Background

Postulated in 1845 by Joseph Bertrand and later proved by Pafnuty Chebyshev in 1850, the Bertrand-Chebyshev theorem is one of the widely used theorems that guarantee the existence of a prime number within a certain interval. The original statement of the theorem is very simple and states the following:

**Theorem 2.1 (Bertrand's postulate)** *For any integer $n > 3$ there exists a prime number $p$ such that*

$$n < p < 2n - 2.$$

Although it has been proved, the theorem is better known as Bertrand's postulate and therefore we use this synonym in the rest of the article. Also, in order to make our calculations simpler, we will use a slightly weaker and probably easier to remember corollary that guarantees the existence of a prime number within the interval $(n, 2n)$ for all integers $n > 1$. It can also be easily generalized to all real numbers $x > 1$. Namely, if $1 < x < 2$ then $p \in (x, 2x)$ for some prime $p$. For $x \geq 2$ we have $x = n + \delta$, where $n = \lfloor x \rfloor$ and $\delta = \{x\}$. Then $n > 1$ is an integer and $0 \leq \delta < 1$ so the interval $(x, 2x) \equiv (n + \delta, 2n + 2\delta)$ contains all the integers from $(n, 2n)$ hence it must also contain a prime number. We use this generalized statement throughout the article and, for the sake of simplicity, we refer to it as Bertrand's postulate.

Many proofs of Bertrand's postulate can be found in the literature. Here, in addition to the historically important first proof given by Pafnuty Chebyshev, we would also like to mention the well known beautiful elementary proof from 1932 given by Paul Erdős. His proof was originally published in [1] and can nowadays be easily found at various sources on the Internet. Interestingly, it was the first published paper of this prolific 20-th century mathematician.

## 3 Some stronger results

The question of the existence of prime numbers has been extensively studied in the past two centuries and several results stronger than Bertrand's postulate have been proved. While providing a review of all of these results falls out of the scope of this article, we list a few of the most relevant refinements of Bertrand's postulate below. For further reading we recommend [2] and [3].

In 1958 Polish mathematician W. Sierpiński postulated that for all $n > 1$ and $k \leq n$ there exists at least one prime number in the closed interval $[kn, (k+1)\,n]$. It is obvious that the statement holds for $k = 1$ as a direct consequence of Bertrand's postulate. Recently, in 2008, M. El Bachraoui gave a proof for $k = 2$ [4]. A simple corollary of this result is that for all $n \geq 1$ the interval $\left(n, \frac{3(n+1)}{2}\right)$ contains at least one prime number. The formal proof of this refinement of Bertrand's postulate can be found in [5]. The case $k = 3$ was proved by A. Loo in 2011 [6] and it leads to a further refinement that guarantees the existence of a prime number in the interval $\left(n, \frac{4(n+2)}{3}\right)$ for all $n \geq 3$.

From the theoretical perspective, both of the refinements above are a consequence of a refinement proved by J. Nagura in 1952. Namely, he proved that for $n \geq 25$ the interval $\left(n, \frac{6n}{5}\right)$ contains at least one prime number [7]. However, his proof relies on more advanced results and concepts from Number Theory and Calculus.

Using the prime number theorem it can also be proved that for any $\epsilon > 0$ there exists $n_0$ such that for all $n > n_0$ the interval $(n, (1+\epsilon)\,n)$ contains at least one prime number. Note that this generalized statement does not give a precise value of $n_0$ and might be unsuitable for finding all solutions of a given equation or solving some similar types of problems.

## 4 Solved Problems

Our primary goal in this article is to demonstrate the applications of the concept of intervals containing at least one prime number and in order to achieve this goal we use Bertrand's postulate. Our key motivation for this choice is the fact that Bertrand's postulate is probably the best known among this group of theorems. On the other hand, as we show here, it is still powerful enough in solving many problems. However, we strongly encourage the reader to simplify the given solutions and strengthen some of the problem statements from this and the next section using refinements of Bertrand's postulate mentioned above.

**Problem 1** *Prove that for any positive integer $k$ there exist at least three different prime numbers having exactly $k$ digits.*

*Solution.* It is trivial to verify that the statement holds for $k = 1$. For $k > 1$, we consider intervals $\left(10^{k-1}, 2 \cdot 10^{k-1}\right)$, $\left(2 \cdot 10^{k-1}, 4 \cdot 10^{k-1}\right)$ and $\left(4 \cdot 10^{k-1}, 8 \cdot 10^{k-1}\right)$. These three intervals are obviously pairwise disjoint and consist only of positive integers having exactly $k$ digits. By Bertrand's postulate each of them contains at leat one prime number, hence the conclusion follows.

Note that $k = 1$ was considered separately since in this case $10^{k-1} = 1$ so Bertrand's postulate can not be applied to the interval $\left(10^{k-1}, 2 \cdot 10^{k-1}\right)$. □

**Problem 2** *Let $p_n$ denote the $n$-th prime number. Prove that*

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n > p_{n+1}^2$$

*holds for all $n \geq 4$.* (Bonse's inequality)

*Solution.* We give a proof based on mathematical induction. For $n = 4$ we have

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 = 2 \cdot 3 \cdot 5 \cdot 7 = 210 > 11^2 = p_5^2$$

so the inequality holds in this case.

Assume now that it holds for all positive integers $n \leq m$. We prove that it also holds for $n = m + 1$. In this case our aim is to prove that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot p_{m+1} > p_{m+2}^2.$$

By Bertrand's postulate there exists a prime number $q$ such that $p_{m+1} < q < 2p_{m+1}$ implying that $p_{m+2} \leq q < 2p_{m+1}$ so in order to complete our proof it is enough to show that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot p_{m+1} > 4p_{m+1}^2.$$

Obviously $p_{m+1} > 4$ so it suffices to prove that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m > p_{m+1}^2.$$

But the last inequality directly follows from the inductive assumption and this completes our proof for the case $n = m + 1$. The induction principle now implies that the given inequality holds for all $n \geq 4$. □

**Problem 3** *If $m$ and $n$ are positive integers prove that*

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n}$$

*is not an integer.*

*Solution.* Let

$$A = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n}.$$

First, observe that for $n \leq m-1$ we obviously have

$$A < m \cdot \frac{1}{m} = 1$$

hence $0 < A < 1$ implying that $A$ is not an integer in this case.

Assume now that $n \geq m$. It is trivial to verify that the problem statement holds for $n = m = 1$, so we may assume that $m+n > 2$. Then $\frac{m+n}{2} > 1$ and using Bertrand's postulate we have that there exists a prime number $p$ such that $\frac{m+n}{2} < p < m+n$. As $2p > m+n$ and $p > \frac{m+n}{2} \geq m$, the prime number $p$ is the only number in the closed interval $[m, m+n]$ that is divisible by $p$.

Now, if we bring all summands in $A$ to a common denominator we obviously get

$$A = \frac{p \cdot B + C}{m \cdot (m+1) \dots (m+n)}$$

where $B \in \mathbb{N}$ and $C = m \cdot (m+1) \dots (p-1) \cdot (p+1) \dots (m+n)$. Analyzing the last fraction, clearly $C$ is not divisible by $p$ hence its numerator is not divisible by $p$. As $p$ is one of the factors in its denominator the whole fraction can not be an integer and this completes our proof. □

**Problem 4** *Prove that the interval* $\left(2^n+1, 2^{n+1}-1\right)$, $n \geq 2$ *contains an integer that can be represented as a sum of n prime numbers.*

(Mathematical Reflections)

*Solution.* Solving this problem is equivalent to finding a set of prime numbers $\{p_1, p_2, \dots, p_n\}$ such that $2^n + 1 < A < 2^{n+1} - 1$, where $A = p_1 + p_2 + \cdots + p_n$. The construction of the set satisfying these conditions is given below.

Define $p_1 = 3$. By Bertrand's postulate there exist prime numbers $p_2, p_3, \dots, p_n$ such that

$$\begin{gathered} 2 < p_2 < 2^2 \\ 2^2 < p_3 < 2^3 \\ \vdots \\ 2^{n-1} < p_n < 2^n. \end{gathered}$$

We prove that $A = p_1 + p_2 + \cdots + p_n$ belongs to the interval $\left(2^n+1, 2^{n+1}-1\right)$.

In order to show that $A > 2^n + 1$, note that

$$A = p_1 + p_2 + \cdots + p_n > 3 + 2 + 2^2 + \cdots + 2^{n-1} =$$

$$2 + \left(1 + 2 + 2^2 + \cdots + 2^{n-1}\right) = 2 + 2^n - 1 = 2^n + 1.$$

Proving that $A < 2^{n+1} - 1$ can be done in an analogous way:

$$A = p_1 + p_2 + \cdots + p_n < 3 + 2^1 + 2^2 + \cdots + 2^n = 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1.$$

□

**Problem 5** *Find all positive integers $m$ such that*

$$1! \cdot 3! \cdot 5! \cdot \dots \cdot (2m-1)! = \left(\frac{m(m+1)}{2}\right)!.$$

(Mediterranean M.C. 2004)

*Solution.* Our main idea in solving this problem is finding a prime number $p$ which divides the right hand side (RHS), but does not divide the left hand side (LHS) of the given equation. In order to find such a prime, observe that for $m > 1$ Bertrand's postulate guarantees the existence of a prime number $p$ such that

$$2m-1 < p < 2(2m-1).$$

Clearly $p \nmid 1! \cdot 3! \cdot 5! \cdot \dots \cdot (2m-1)!$ and $p \le 4m-3$.

Now, if $4m-3 \le \frac{m(m+1)}{2}$, which is equivalent to $m \ge 6$, we have $p \le \frac{m(m+1)}{2}$. This implies that $p \mid \left(\frac{m(m+1)}{2}\right)!$ so in this case $p$ divides RHS, but does not divide LHS. Consequently, there is no solution for $m \ge 6$.

Therefore it remains to check cases where $m < 6$. Direct validation shows that $m = 1$ is the only solution. □

**Problem 6** *For an integer $n > 3$ define $n?$ as a product of all prime numbers less than $n$. Find all integers $n > 3$ such that*

$$n? = 2n + 16.$$

(Russia, 2007)

*Solution.* The LHS of given equation is divisible by 2, but not by 4. This implies that $n$ is an odd number (otherwise the RHS is divisible by 4). Let $n = 2k+1$ for some integer $k \ge 2$. Then our equation becomes

$$(2k+1)? = 2(2k+9).$$

By Bertrand's postulate there exists a prime number $p$ such that $k < p < 2k$. Then $p \mid (2k+1)?$ which implies $p \mid 2(2k+9)$. Since $p \ge k+1 \ge 3$, $p$ must be an odd prime so $p \mid (2k+9)$. Now, note that $p < 2k$ implies $p < 2k+9$ so $p$ is a proper divisor of $2k+9$. On the other hand, since $2k+9$ is odd, it is not divisble by 2 so its smallest divisor is greater than or equal to 3. As a consequence, its greatest proper divisor is not greater than $\frac{2k+9}{3}$. Since $p$ is one of its proper divisors we conclude that $p \le \frac{2k+9}{3}$. This leads us to the following inequality

$$k+1 \le p \le \frac{2k+9}{3}$$

that implies $k \le 6$. This is equivalent to $n \le 13$ so in order to complete our solution it remains to inspect for the values of $n$ such that $4 \le n \le 13$ and $n? = 2n+16$. For $8 \le n \le 13$ we have $n? \ge 2 \cdot 3 \cdot 5 \cdot 7 > 2 \cdot 13 + 16 \ge 2n+16$. Therefore we do not have solutions for $n \ge 8$. Direct verification for $n = 4, 5, 6, 7$ shows that $n = 7$ is the only solution of the given equation. □

**Problem 7** *Find all positive integers $n$ for which the number of all positive divisors of the number* $\operatorname{lcm}(1, 2, \ldots, n)$ *is equal to* $2^k$ *for some non-negative integer* $k$*.*

(Estonia, IMO TST 2004)

*Solution.* Assume that we are given a fixed $n$. Our main idea in solving this problem is finding a prime number $p$ such that $p^2 < n < p^3$. Namely, if such prime $p$ exists then $\operatorname{lcm}(1, 2, \ldots, n) = p^2 \cdot A$, where $A$ is some positive integer and $\gcd(A, p) = 1$. This directly implies that the number of divisors of $\operatorname{lcm}(1, 2, \ldots, n)$ is divisible by 3 and therefore it can not be equal to $2^k$ for some non-negative integer $k$.

In order to find such a prime $p$, observe that for $n > 4$ by Bertrand's postulate there exists a prime number $p$ such that $\frac{\sqrt{n}}{2} < p < \sqrt{n}$. This directly implies $p^2 < n$ and it remains to discuss whether $n < p^3$ holds. Note that, by the choice of $p$, we have

$$p^3 > \left(\frac{\sqrt{n}}{2}\right)^3 = n \cdot \frac{\sqrt{n}}{8}.$$

It is now obvious that $p^3 > n$ holds for all $n \geq 64$ implying that there is no solution for any such $n$. We may now assume that in the rest of our solution $n < 64$.

Although the brute force approach is already applicable at this point, we can avoid it for most of the cases by providing an exact value of $p$ depending on $n$. It is enough to observe that $p = 5$ works for all $n$ such that $25 < n < 64$. Similarly, if $9 < n \leq 25$ we have $3^2 < n < 3^3$ leading to the choice $p = 3$. For $n \leq 9$ direct verification shows that $n = 1, 2, 3, 8$ are the only solutions. □

# 5 Problems for Self-study

**Problem 1** *Let $p_n$ denote the $n$-th prime number (i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$). Prove that $p_n \leq 2^n$.*

**Problem 2** *Find all integers $n > 1$ and $m > 1$ such that*

$$1! \cdot 3! \cdot 5! \cdots (2n-1)! = m!.$$

(American Mathematical Monthly)

**Problem 3** *Determine all triplets of positive integers $(k, m, n)$ such that*

$$n! = m^k.$$

(Singapore IMO training)

**Problem 4** *Find the greatest integer that cannot be written as a sum of distinct prime numbers.*

(Mathematical Reflections)

**Problem 5** *Prove that the number*

$$a_n = \sum_{k=n}^{2n} \frac{(2k+1)^n}{k}$$

*is not an integer for any positive integer $n$.*

(Austria, Regional Competition, 2008)

**Problem 6** *Prove that for each positive integer $k$ the integers $\{1, 2, \ldots, 2k\}$ can be arranged into $k$ disjoint pairs so that the sums of the elements in each pair is prime.*

***Comment:*** *For a proof and some interesting applications of this result see [8].*

**Problem 7** *Determine all triplets $(a, b, c)$ of positive integers that satisfy*

$$a! + b^b = c!.$$

(Crux Mathematicorum)

**Problem 8** *Let $n$ be a natural number such that $m$ divides $n$ for each positive integer $m < \sqrt{n}$. Prove that $n < 49$.*

(Albania NMO 2005)

***Hint***: We suggest using Nagura's refinement of Bertrand's postulate for the remaining three problems given below.

**Problem 9** *Find the smallest positive integer $n_0$ such that for all integers $n > n_0$ the interval $(n, 2n)$ contains at least three prime numbers.*

**Problem 10** *Let $p_n$ denote the $n$-th prime number. Prove that*

$$\frac{5}{2} \geq \frac{p_{n-1} + p_{n+1}}{p_n} \geq \frac{3}{2}.$$

*(www.artofproblemsolving.com)*

**Problem 11** *For each positive integer $n$ , determine the least integer $m$ such that*

$$\operatorname{lcm}\{1, 2, \ldots, m\} = \operatorname{lcm}\{1, 2, \ldots, n\}.$$

(American Mathematical Monthly)

# References

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