Crux Mathematicorum

VOLUME 42, NO. 4 April / Avril 2016

IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

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Crux Mathematicorum with Mathematical Mayhem

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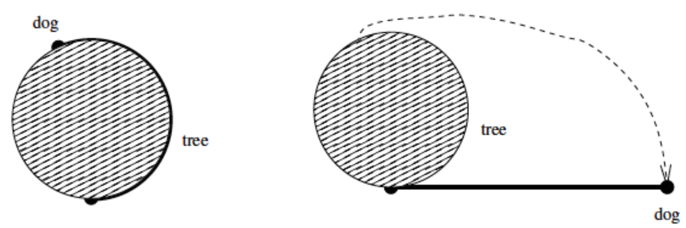
EDITORIAL

Canadians live in the world's most educated country: according to the Organization for Economic Cooperation and Development, Canada tops the charts with 54% of population having tertiary education. And this is up from 40% in year 2000. Of course, highly educated immigrants contribute to these figures, but it is not the only explanation. Canada also confers more bachelors degrees per capita than any other country. So no wonder that this is where you will find some of the most avid problem solvers and many sources of interesting problems to puzzle over.

From East to West, nearly every university in the country runs undergraduate contests for their students. To entice the unsuspecting and the amenable undergraduates into mathematics, these contests often offer unusual problems. Here are two examples from two universities on either coast.

1. A dog is tied to a tree trunk of radius 1 by a rope of length 10 attached at a fixed point on the trunk of the tree. The rope is initially taut and fully wound around the trunk. The dog runs around the tree unwinding the rope and keeping the rope taut until the rope is tangential to the tree trunk. This is illustrated (not to scale) in the figures below: the left panel shows the initial position and the right panel shows the final position, with the thick line representing the rope and the dashed line representing the path of the dog.



What is the total distance run by the dog? As indicated in the figure, you should assume that all motion takes place in the horizontal plane. (University of Victoria Mathematics Competition 2013.)

2. Let $ABCD$ be an isosceles trapezoid with $|AB| = |CD| = F_n$, $|BC| = F_{n-1}$ and $|AD| = F_{n+1}$, where F_k are the Fibonacci numbers $F_1 = F_2 = 1, F_{k+1} = F_k + F_{k-1}$. Find the area of the trapezoid and express it in terms of a single Fibonacci number. (Memorial University of Newfoundland Undergraduate Mathematics Competition 2009.)

Although this country covers nearly 10 million square kilometres, you can always find a mathematical inspiration nearby. Participate in these events and contribute to them to keep generations of problem solvers entertained.

Kseniya Garaschuk

THE CONTEST CORNER

No. 44

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2017***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction remercie André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC216. Étant donné un ensemble de trois nombres, on crée un deuxième ensemble en remplaçant chaque nombre de l'ensemble initial par la somme des deux autres nombres de l'ensemble initial. Par exemple, à partir de l'ensemble $\{3, 4, 6\}$, on obtient l'ensemble $\{10, 9, 7\}$ et à partir de ce deuxième ensemble, on obtient l'ensemble $\{16, 17, 19\}$. En partant de l'ensemble $\{20, 1, 3\}$, on applique la procédure 2014 fois. Quelle est la plus grande différence entre n'importe quels deux nombres de l'ensemble qui en résulte?

CC217. L'hypoténuse AB d'un triangle rectangle ABC est coupée en trois parties égales par les points M et N. Sachant que $CM^2 + CN^2 = k \cdot AB^2$, quelle est la valeur de k?

CC218. Résoudre ce système d'équations:

$$3^{\ln x} = 4^{\ln y}$$
$$(4x)^{\ln 4} = (3y)^{\ln 3}$$

CC219. Un morceau de bois, qui a la forme d'un prisme droit à base rectangulaire, mesure 4 sur 5 sur 6. On recouvre le solide de peinture verte, puis on le coupe en cubes de dimensions 1 sur 1 sur 1. Déterminer le rapport du nombre de petits cubes ayant exactement deux faces vertes au nombre de petits cubes ayant exactement trois faces vertes.

CC220. On tire au hasard deux nombres x et y de l'intervalle fermé $[0, 2]$. Quelle est la probabilité pour que $x + y > 1$?

CC216. Starting with a list of three numbers, the "changesum" procedure creates a new list by replacing each number by the sum of the other two. For example, from $\{3, 4, 6\}$ "changesum" gives $\{10, 9, 7\}$ and a new "changesum" leads to $\{16, 17, 19\}$. If we begin with $\{20, 1, 3\}$, what is the maximum difference between two numbers of the list after 2014 consecutive "changesums"?

CC217. A right triangle ABC has its hypotenuse AB trisected at M and N. If $CM^2 + CN^2 = k \cdot AB^2$, then what is the value of k?

CC218. Solve the following system of equations:

$$\begin{aligned} 3^{\ln x} &= 4^{\ln y}, \\ (4x)^{\ln 4} &= (3y)^{\ln 3}. \end{aligned}$$

CC219. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. Find the ratio of the number of cubes with exactly two green faces to the number of cubes with exactly three green faces.

CC220. Two random numbers x and y are drawn independently from the closed interval $[0, 2]$. What is the probability that $x + y > 1$?

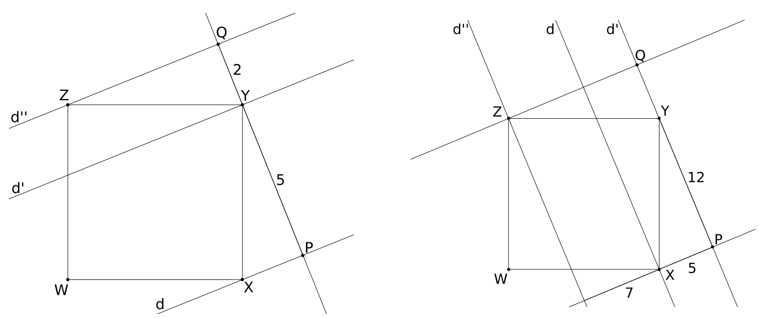
CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(4), p. 143–144.

CC166. Let $WXYZ$ be a square. Three parallel lines d, d' and d'' pass respectively through X, Y and Z. The distance between d and d' is 5 and the distance between d and d'' is 7. What is the area of the square?

Originally question 1 of the 2015 Midi Belgian mathematics contest.

The question as posed had two possible solutions. We received eight submissions, all of which correctly solved one of the two cases. The two cases are shown in the figure. We present the solution of John Heuver for the first case.



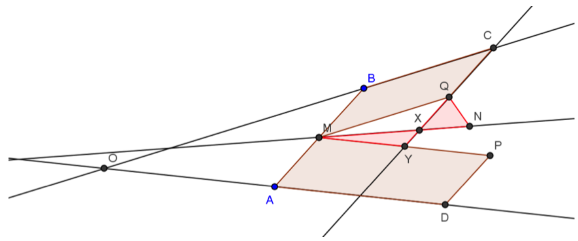
Draw a perpendicular line through Y meeting d' and d'' in P and Q respectively, then $|PY| = 5$ and $|QY| = 2$. Right angled triangles XPY and YQZ are congruent on account of alternating and complementary angles while $WXYZ$ is a square. Hence $|XP| = 2$ and thus from the triangle XPY we obtain $|XY| = \sqrt{29}$. Thus the area of $WXYZ$ is 29.

[*Editor's comment.* The area in the second case is 169.]

CC167. The lines BC and AD intersect at O, with both B and C on the same side of O, and the same goes for A and D. Among other properties, we have $|BC| = |AD|$, $2|OC| = 3|OB|$ and $|OD| = 2|OA|$. Points M and N are the respective middle points of segments $[AB]$ and $[CD]$. Quadrilaterals $ADPM$ and $BCQM$ are parallelograms. The line CQ cuts MN and MP respectively at X and Y. Show that triangles MXY and QXN have the same area.

Originally question 3 of the 2015 Midi Belgian mathematics contest.

We received four correct solutions. We present the solution by Titu Zvonaru.



Let the line MP intersect OC at R. Since M is the midpoint of AB and $MR \parallel AO$ by the definition of P, we get that $|RB| = |OB|/2$. Combining this with the information that $2|OC| = 3|OB|$ we get $|OR| = |RB| = |BC|$.

Let F be the intersection of the line CQ with OD. Since $CF \parallel BA$, we have that $|BC|/|OB| = |AF|/|OA|$. From the conditions given in the question $|BC|/|OB| = 1/2$ and $|OA| = |AD|$, so it follows that F is the midpoint of AD.

Consider $\triangle COF$. Since RY extends MY, we have $RY \parallel OF$, whence

$$\frac{|YC|}{|YF|} = \frac{|CR|}{|OR|} = 2.$$

Now consider the points A, Y and N which are on the extended sides of $\triangle CFD$ (by construction, N is the midpoint of CD). We have

$$\frac{|FA|}{|AD|} \cdot \frac{|DN|}{|NC|} \cdot \frac{|CY|}{|YF|} = \frac{1}{2} \cdot 1 \cdot 2 = 1.$$

By the converse Menelaus Theorem it follows that the points A, Y and N are collinear. Since A is the midpoint of OD and N is the midpoint of CD we have $AN \parallel OC$, which in turn, since $MQCB$ is a parallelogram and A, Y, N are collinear, means that $YN \parallel MQ$. This implies that $[MYQ] = [MNQ]$, which allows us to conclude that

$$[MXY] = [MYQ] - [MXQ] = [MNQ] - [MXQ] = [QXN],$$

as desired.

CC168. Six students from different European countries participate in an Erasmus course together. Each student speaks exactly two languages. Angela speaks German and English; Ulrich, German and Spanish; Carine, French and Spanish; Dieter, German and French; Pierre, French and English and Rocio Spanish and English. If we choose 2 people at random, what is the probability that they speak a common language?

Originally question 24 of the 2015 Primavera Mathematics Contest of Spain.

We received five submissions of which four were correct and complete. We present the solution by Henry Ricardo.

There are $6(5)/2 = 15$ ways to choose a pair of students at random. It is easy to see that only three of the 15 possible pairs *do not* share at least one language in common – Angela-Carine, Ulrich-Pierre, and Dieter-Rocio. Thus the desired probability is

$$1 - 3/15 = 4/5.$$

CC169. What is the value of base b if

$$\log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} = 110.$$

Originally problem 17 of the 2015 Primavera Mathematics Contest of Spain.

We received eight complete and correct solutions, and present a composite of those solutions here.

Since $\log_b 10^k = k \log_b 10$ we have

$$\begin{aligned} \log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} &= 110 \\ (1 + 2 + \cdots + 10) \log_b 10 &= 110 \\ \frac{10 \cdot 11}{2} \log_b 10 &= 110 \\ \log_b 10 &= 2 \end{aligned}$$

so $b^2 = 10$, which gives $b = \sqrt{10}$ (discarding the negative option because a base cannot be negative).

CC170. The sum of 35 integers is S. We change 2 digits of one of the integers and the new sum is T. The difference $S - T$ is always divisible by which of the 5 numbers $2, 5, 7, 9$ or 11?

Originally problem 15 of the 2015 Primavera Mathematics Contest of Spain.

We received six correct solutions. We present the solution of Hannes Geupel.

The difference $S - T$ is always divisible by 9. We only need to look at the number that gets changed in S and T, because the other numbers get subtracted by themselves in $S - T$. For example let this number be 21. $21 - 12 = 9$. 9 is not divisible by 2, 5, 7 or 11, so only the divisor 9 is left.

Now we prove that $S - T$ is always divisible by 9. Let N be a 2-digit number which is the one integer of S which gets changed. Let the first digit of N be a and the second digit be b. So $S - T = (10a + b) - (10b + a) = 9(a - b)$. Obviously, $9 \mid 9(a - b)$. In the general case, the two changed digits are $10^m a$ and $10^n b$ (the m^{th} and n^{th} digits are a and b respectively).

$S - T = (10^m a + 10^n b) - (10^m b + 10^n a) = (10^m - 10^n)(a - b)$. It is not hard to see that $10^m \equiv 1 \pmod 9$ and $10^n \equiv 1 \pmod 9$ so it follows that $10^m - 10^n \equiv 0 \pmod 9$. Hence $(10^m - 10^n)(a - b) \equiv 0 \pmod 9$ or rather, $9 | (10^m - 10^n)(a - b) = S - T$.

THE OLYMPIAD CORNER

No. 342

Carmen Bruni

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2017***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction souhaite remercier André Ladouceur, Ottawa, d'avoir traduit les problèmes.

OC276. Soit m et n des entiers strictement positifs. Sachant que le nombre

$$k = \frac{(m+n)^2}{4m(m-n)^2+4}$$

est un entier, démontrer que k est un carré parfait.

OC277. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ qui vérifient

$$f(x^2 + yf(x)) = xf(x+y)$$

pour tous réels x, y.

OC278. Soit n un entier supérieur ou égal à 4. Déterminer toutes les permutations $\{x_1, x_2, \ldots, x_n\}$ de $\{1, 2, \ldots, n\}$ pour lesquelles $x_i < x_{i+2}$ lorsque $1 \leq i \leq n-2$ et $x_i < x_{i+3}$ lorsque $1 \leq i \leq n-3$.

OC279. Soit ABC un triangle acutangle et O le centre de son cercle circonscrit. Soit P et Q des points sur les côtés respectifs AB et AC tels que

$$BP \cdot CQ = AP \cdot AQ.$$

Soit I un cercle dont le centre est situé sur la hauteur du triangle ABC abaissée au point A et qui passe aux points A, P et Q. Démontrer que I est tangent au cercle circonscrit au triangle BOC.

OC280. Soit $g(n)$ le plus grand commun diviseur de n et 2015. Déterminer le nombre de triplets (a, b, c) qui satisfont aux deux conditions suivantes:

1. $a, b, c \in \{1, 2, \ldots, 2015\}$ et
2. $g(a), g(b), g(c), g(a+b), g(b+c), g(c+a), g(a+b+c)$ sont distincts.

. . . .

OC276. Let m and n be positive integers. If the number

$$k = \frac{(m+n)^2}{4m(m-n)^2+4}$$

is an integer, prove that k is a perfect square.

OC277. Find all real functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2+yf(x)) = xf(x+y)$.

OC278. Find all possible permutations $\{x_1, x_2, \ldots, x_n\}$ of $\{1, 2, ..., n\}$ so that when $1 \leq i \leq n-2$ then we have $x_i < x_{i+2}$ and when $1 \leq i \leq n-3$ then we have $x_i < x_{i+3}$. Here $n \geq 4$.

OC279. Let ABC be an acute-angled triangle with circumcenter O. Let I be a circle with centre on the altitude from A in ABC, passing through vertex A and points P and Q on sides AB and AC. Assume that

$$BP \cdot CQ = AP \cdot AQ.$$

Prove that I is tangent to the circumcircle of triangle BOC.

OC280. Let $g(n)$ be the greatest common divisor of n and 2015. Find the number of triples (a, b, c) which satisfy the following two conditions:

1. $a, b, c \in 1, 2, ..., 2015$;
2. $g(a), g(b), g(c), g(a+b), g(b+c), g(c+a), g(a+b+c)$ are pairwise distinct.

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(2), p. 55–56.

OC216. Let $p = n^2 + 1$ be a given prime number. Find the set of integer solutions to the equation below:

$$x^2 - (n^2 + 1)y^2 = n^2.$$

Originally problem 4 from the number theory portion of the third round of the 2013 Iranian National Mathematical Olympiad.

We received 2 correct submissions. We present the solution by Oliver Geupel.

We will show that the solutions for $p = 2$ are $x + y\sqrt{2} = \pm(3 + 2\sqrt{2})^k$, where $k \in \mathbb{Z}$. We will prove that the solutions for $p > 2$ are

$$\begin{aligned} x + y\sqrt{p} \in \{ &\pm (2n^2 + 1 + 2n\sqrt{p})^k n, \\ &\pm (2n^2 + 1 + 2n\sqrt{p})^k (n^2 - n + 1 + (n-1)\sqrt{p}), \\ &\pm (2n^2 + 1 + 2n\sqrt{p})^k (n^2 + n + 1 + (n+1)\sqrt{p}) \mid k \in \mathbb{Z}\}. \end{aligned}$$

For $z = x + y\sqrt{p} \in \mathbb{Z}[\sqrt{p}]$, denote $N(z) = x^2 - py^2$. We recapitulate the following well-known facts on Pell-like equations (check, e.g., the article *Pell's Equations* by Dušan Dukić on the website of the IMO compendium, retrieved March 25, 2016 from `http://imomath.com/index.php?options=615`):

(1) If z_1 is the fundamental solution (which must exist) of the equation $N(z) = 1$, i.e., the minimal element of $\mathbb{Z}[\sqrt{p}]$ with $z > 1$ and $N(z) = 1$, then all the solutions $z \in \mathbb{Z}[\sqrt{p}]$ are given by $z = \pm z_1^k$, $k \in \mathbb{Z}$.

(2) The fundamental solution of the equation $x^2 - 2y^2 = 1$ is $3 + 2\sqrt{2}$.

(3) For $a \in \mathbb{Z}$, every solution of the equation $N(z) = a$ has the form $z = \pm z_1^k z_a$ ($k \in \mathbb{Z}$) where z_1 is the fundamental solution of the equation $N(z) = 1$, and $z_a = x_a + y_a\sqrt{p}$ is a solution of $N(z) = a$ with $1 \le z_a \le z_1$. Also

$$|x_a| \le \frac{z_1 + 1}{2\sqrt{z_1}}\sqrt{|a|}.$$

The result for $p = 2$ follows from (1) and (2). It remains to consider $p > 2$.

Let $z_1 = x_1 + y_1\sqrt{p}$ be the fundamental solution of the equation $N(z) = 1$. We obtain $y_1^2 = (x_1 - 1)(x_1 + 1)/p$, that is, p is a divisor of either $x_1 - 1$ or $x_1 + 1$. We check the small values for x_1 in succession. If $x_1 = p - 1$ then $y_1^2 + 1 = n^2$, which is impossible. If $x_1 = p + 1$ then $y_1^2 = n^2 + 3$ which is impossible for $n > 1$. Trying $x_1 = 2p - 1$, we obtain $z_1 = 2n^2 - 1 + 2n\sqrt{p}$.

To complete the work, by (3) it is enough to find the solutions $z_a = x_a + y_a\sqrt{p}$ of $N(z_a) = a$ with $a = n^2$ and

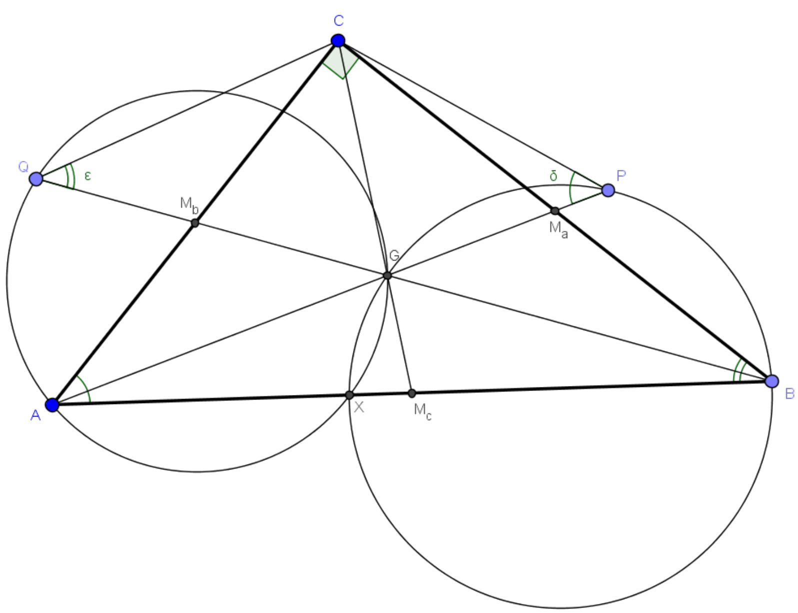
$$x_a < \frac{2n^2 + 2n\sqrt{n^2+1}}{2\sqrt{2n^2 - 1 + 2n\sqrt{n^2+1}}} \cdot n < 2n^2.$$

From $N(z_a) = a$ we have $y_a^2 = (x_a - n)(x_a + n)/p$, that is, p divides either $x_a - n$ or $x_a + n$. We check the small values for x_a in succession. Trying $x_a = n$, we find $z_a = n$. Putting $x_a = p - n$, we obtain $z_a = n^2 - n + 1 + (n-1)\sqrt{p}$. For $x_a = p + n$, we get $z_a = n^2 + n + 1 + (n+1)\sqrt{p}$. Trying $x_a = 2p - n$, we obtain $y_a^2 = 4(p - n)$. Hence, $p - n = n^2 + 1 - n$ is a perfect square, say m^2. We obtain $(n-1)^2 < m^2 < n^2$, a contradiction. The solution is complete.

OC217. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90°$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

Originally problem 3 of the 2013 Canadian Mathematical Olympiad.

We received 3 correct submissions. We present the solution by Andrea Fanchini.



We use barycentric coordinates and the usual Conway's notations with reference to triangle ABC. With $\angle BCA = 90°$, we have $S_C = 0$ so $S_A = b^2$, $S_B = a^2$, $c^2 = a^2 + b^2$. Then a point P on ray AG has coordinates $P(u : 1 : 1)$ where u is a parameter. Now the oriented angle δ (with $0 \leq \delta \leq \pi$) between two lines

$d_i \equiv p_i x + q_i y + r_i z = 0 (i = 1, 2)$, is given from

$$S_\delta = S \cot \delta = \frac{S_A(q_1 - r_1)(q_2 - r_2) + S_B(r_1 - p_1)(r_2 - p_2) + S_C(p_1 - q_1)(p_2 - q_2)}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

Therefore using the above, the angle between the line $CP \equiv x - uy = 0$ and the median $AP \equiv y - z = 0$, is

$$S_\delta = \frac{a^2 - 2b^2 u}{u + 2}.$$

But $\angle CPA = \angle CAB$, so

$$S_\delta = S_A = b^2 \quad \Rightarrow \quad u = \frac{a^2 - 2b^2}{3b^2} \quad \Rightarrow \quad P(a^2 - 2b^2 : 3b^2 : 3b^2).$$

Now the equation of a circle is $a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0$, so to find the equation of the circumcircle of $\triangle BPG$ we have to put in the coordinates of $B(0:1:0)$, $G(1:1:1)$, $P(a^2 - 2b^2 : 3b^2 : 3b^2)$ and solving the system, we have $p = b^2, q = 0, r = \frac{2a^2 - b^2}{3}$. Therefore, the intersection between the circumcircle BPG and the side AB gives the point X

$$\begin{cases} a^2yz + b^2zx + c^2xy - (x+y+z)(b^2x + \frac{2a^2-b^2}{3}z) = 0 \\ z = 0 \end{cases} \quad \Rightarrow \quad X(a^2 : b^2 : 0).$$

In the same way, we have $Q(1 : v : 1)$ and the angle between the median $BQ \equiv x - z = 0$ and the line $CQ \equiv vx - y = 0$, is

$$S_\epsilon = \frac{b^2 - 2a^2 v}{v + 2}.$$

But $\angle CQB = \angle ABC$, so

$$S_\epsilon = S_B = a^2 \quad \Rightarrow \quad v = \frac{b^2 - 2a^2}{3a^2} \quad \Rightarrow \quad Q(3a^2 : b^2 - 2a^2 : 3a^2).$$

To find the equation of the circumcircle of $\triangle AQG$, we have to put in the coordinates of $A(1:0:0)$, $G(1:1:1)$, $Q(3a^2 : b^2 - 2a^2 : 3a^2)$ and solving the system, we have $p = 0, q = a^2, r = \frac{2b^2 - a^2}{3}$. Therefore, the intersection between the circumcircle AQG and the side AB gives the point X'

$$\begin{cases} a^2yz + b^2zx + c^2xy - (x+y+z)(a^2y + \frac{2b^2-a^2}{3}z) = 0 \\ z = 0 \end{cases} \quad \Rightarrow \quad X'(a^2 : b^2 : 0).$$

So we have $X \equiv X'$, as required.

Note: this point is the intersection of the symmedian through vertex C and the side AB. The three symmedians concur at the Symmedian Point $K(a^2 : b^2 : c^2)$, well known also as Lemoine point.

OC218. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$

for all positive integers m, n.

Originally problem 5 from day 2 of the 2013 Korean National Olympiad.

We have a partial solution submitted by Konstantine Zelator which we include (and complete) here.

We claim that the only solutions are $f(m) = km$ for any $k \in \mathbb{N}$. First, this function is a solution since

$$\operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n)) = \frac{mn}{\gcd(m, n)} \cdot \gcd(km, kn) = kmn = f(mn).$$

Now, let's call the given relation $R_{m,n}$ and suppose $f(1) = c$. Then, $R_{m,1}$ for any $m \in \mathbb{N}$ gives

$$f(m) = m \gcd(f(m), c).$$

Now, by the relation $R_{cm,1}$ for any $m \in \mathbb{N}$ (and using the above), we see that

$$f(cm) = cm \cdot \gcd(f(cm), c) = cm \cdot \gcd(c \cdot \gcd(f(m), c), c) = c^2 m.$$

Next, the relation $R_{m,cn}$ for any $m, n \in \mathbb{N}$ reveals with the above two facts that

$$c^2 mn = f(mcn) = \operatorname{lcm}(m, cn) \cdot \gcd(f(m), f(cn)) = \frac{cmn}{\gcd(m, cn)} \cdot \gcd(f(m), c^2 n)$$

and simplifying yields

$$c \gcd(m, cn) = \gcd(f(m), c^2 n).$$

Therefore, $c \mid f(m)$. Hence, from the first derived relation above, we see that

$$f(m) = m \gcd(f(m), c) = mc$$

completing the proof.

OC219. Given positive integers m and n, prove that there is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten.

Originally problem 5 from day 2 of the 2013 USAMO.

We present the solution by Oliver Geupel. There were no other submissions.

Without loss of generality assume $m < n$. We consider the cases $\gcd(m, 10) = 1$ and $\gcd(m, 10) > 1$ in succession.

First, consider the case $\gcd(m, 10) = 1$. We have $\gcd\left(10^{n^2} n - m, 10\right) = 1$. Hence, there exists an integer $a > 3n^2$ such that $10^a \equiv 1 \left(\bmod 10^{n^2} n - m\right)$. Fix such an integer a and put $b = a - 2n^2$. As a consequence, we have $b > n^2$. Note that $m \equiv 10^{n^2} n \pmod{10^{n^2}n - m}$. Hence,

$$10^{n^2+b}m \equiv 10^{2n^2+b}n \equiv n \pmod{10^{n^2}n - m},$$

that is, the number $d = \dfrac{10^{n^2+b}m - n}{10^{n^2}n - m}$ is an integer.

We are going to show that

$$c = 10^{3b} + d$$

has the desired property.

By direct computation, we have

$$cm = 10^{3b}m + 10^{n^2} \cdot \frac{10^b m^2 - n^2}{10^{n^2}n - m} + n, \qquad cn = 10^{3b}n + 10^b m + \frac{10^b m^2 - n^2}{10^{n^2}n - m}.$$

Thus,

$$f = \frac{10^b m^2 - n^2}{10^{n^2}n - m}$$

is an integer. We have $0 < f < 10^b$. Consequently, the decimal expansion of cm is the concatenation of the decimal expansions of the three numbers m, f, and n, padded with some additional zeros. Also, the decimal representation of cn is the concatenation of the decimal representations of n, m, and f, padded with some extra zeros. This completes the proof in the case $\gcd(m, 10) = 1$.

It remains to examine the case $\gcd(m, 10) > 1$. Let $m = 2^h 5^j k$ with nonnegative integers h, j, and k where $\gcd(k, 10) = 1$. For positive integers x and y, we write $x \sim y$ if the decimal representations of x and y have the same number of occurrences of each nonzero digit. We know that there is a positive integer c_0 such that $c_0 k \sim c_0 2^j 5^h n$. We obtain

$$2^j 5^h c_0 m = c_0 10^{h+j} k \sim c_o k \sim 2^j 5^h c_0 n.$$

Consequently, the number

$$c = 2^j 5^h c_0$$

has the required property for m and n.

OC220. Let $A_1A_2 \ldots A_8$ be a convex octagon such that all of its sides are equal and its opposite sides are parallel. For each $i = 1, \ldots, 8$, define B_i as the intersection between segments A_iA_{i+4} and $A_{i-1}A_{i+1}$, where $A_{j+8} = A_j$ and $B_{j+8} = B_j$ for all j. Show some number i, amongst 1, 2, 3, and 4 satisfies

$$\frac{A_iA_{i+4}}{B_iB_{i+4}} \le \frac{3}{2}.$$

Originally problem 6 of the 2013 Mexican National Olympiad.

No submitted solutions.

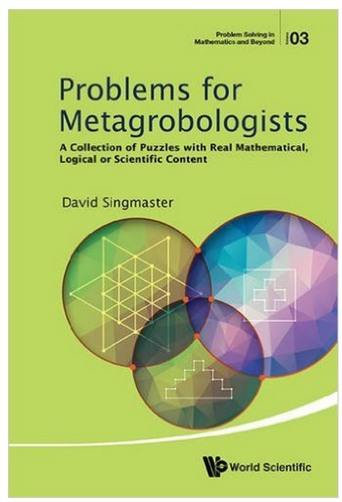
BOOK REVIEWS

Robert Bilinski

Problems for Metagrobologists: A Collection of Puzzles With Real Mathematical, Logical or Scientific Content by David Singmaster
ISBN 9814663638, 246 pages
Published by World Scientific Publishing Company, 2016.

Reviewed by ***Robert Bilinski****, Collège Montmorency.*

David Singmaster is an American/British mathematician known for studying the Rubik's cube and devising the notation to do so. He specializes in number theory, combinatorics and recreational mathematics. This book is an anthology of published and unpublished problems.The title is based on a 16th century English verb "metagrobolize", meaning to puzzle over. The subtitle, "A collection of puzzles with real mathematical, logical or scientific content", is a claim we will explore later in the review.



The book is split into two parts. The first contains 221 problems spread out over 100 pages and organized into the following 15 chapters : General arithmetic puzzles, Properties of digits, Magic figures, Monetary problems, Diophantine recreations, Alphametics, Sequence puzzles, Logic puzzles, Geometrical puzzles, Geographic problems, Calendrical problems, Clock problems, Physical problems, Combinatorial problems and Some verbal puzzles. The second part of the book contains complete solutions to the problems. They are written in an informal manner, more as sketches of solutions or proofs rather than rigorous proofs.

The problems in the book seem to have been written to avoid mathematical notation as much as possible, ergo be as textual as possible, sometimes at the cost of brevity and clarity. But that is the cost of publishing in mass market media (BBC, CBC, Focus, L.A. Times, etc). This probably gives the slightly wrong impression that the puzzles only need elementary mathematics to be solved. The more complex solutions need a bit of algebra, some knowledge of geometry, a smidgen of sequences or some knowledge of combinatorics. A few problems end with an open question widening its scope or generalizing in some fashion, for example :

> 74. Doubling up.
> Jessica and Sophie were playing poker with matchsticks. Each time, they bet as much as possible, namely as much as the poorer one had. Jessica won first, then Sophie, then alternatively through six hands all together. At this point, they were amazed to notice that they had the

> same number of matchsticks. What are the least number of matchsticks they could have had at the beginning? [If that's too easy, solve the problem for n hands.]

A few of the problems seem like rewritings of classics with a personal twist by the author, but a good part are original problems. In fact, the cultural clash which Singmaster lived through when he moved to London in the 1970's is at the heart of some of the problems proposed in his book. For example,

> 60. No Change.
> I recently wanted to pay someone for an item that cost less than £5, so I offered him a £5 note. He said he couldn't give me the right change. So I jestingly offered him two £5 notes and was surprised when he said that would do nicely. How can this happen? Can this happen in America?

All in all, having been hooked on problem solving for quite a while, I found quite a few of the problems to be original or at the least presented in an original manner. I do admit to having had some difficulty with the wording of some of the problems. It is this book's weakness and forte at the same time, since it will appeal to a wider audience, namely the non-scientific puzzlers in your midst who seek out challenging entertainment. As such, I definitely recommend you consider this book as a gift idea. Naturally, you shouldn't deprive yourself of an opportunity to amuse yourself either. Good reading!

Applications of Bertrand's postulate and its extensions in Math Olympiad style problems

Salem Malikić

1 Introduction

Prime numbers are one of the fundamental entities in Number Theory. The guarantee of the existence of a prime number within a certain interval can be helpful in solving several types of problems. The Bertrand-Chebyshev theorem, also known as Bertrand's postulate, can be very useful in this context. Here we present several solved examples where it can be successfully used. A particular emphasis is placed on Math olympiad-style problems and therefore the article is based solely on elementary techniques. In addition to solved examples, this work also contains a brief historical and theoretical background, a list of some stronger results as well as a set of problems for self-study.

2 Historical and Theoretical Background

Postulated in 1845 by Joseph Bertrand and later proved by Pafnuty Chebyshev in 1850, the Bertrand-Chebyshev theorem is one of the widely used theorems that guarantee the existence of a prime number within a certain interval. The original statement of the theorem is very simple and states the following:

Theorem 2.1 (Bertrand's postulate) *For any integer $n > 3$ there exists a prime number p such that*

$$n < p < 2n - 2.$$

Although it has been proved, the theorem is better known as Bertrand's postulate and therefore we use this synonym in the rest of the article. Also, in order to make our calculations simpler, we will use a slightly weaker and probably easier to remember corollary that guarantees the existence of a prime number within the interval $(n, 2n)$ for all integers $n > 1$. It can also be easily generalized to all real numbers $x > 1$. Namely, if $1 < x < 2$ then $p \in (x, 2x)$ for some prime p. For $x \geq 2$ we have $x = n + \delta$, where $n = \lfloor x \rfloor$ and $\delta = \{x\}$. Then $n > 1$ is an integer and $0 \leq \delta < 1$ so the interval $(x, 2x) \equiv (n + \delta, 2n + 2\delta)$ contains all the integers from $(n, 2n)$ hence it must also contain a prime number. We use this generalized statement throughout the article and, for the sake of simplicity, we refer to it as Bertrand's postulate.

Many proofs of Bertrand's postulate can be found in the literature. Here, in addition to the historically important first proof given by Pafnuty Chebyshev, we would also like to mention the well known beautiful elementary proof from 1932 given by Paul Erdős. His proof was originally published in [1] and can nowadays be easily found at various sources on the Internet. Interestingly, it was the first published paper of this prolific 20-th century mathematician.

3 Some stronger results

The question of the existence of prime numbers has been extensively studied in the past two centuries and several results stronger than Bertrand's postulate have been proved. While providing a review of all of these results falls out of the scope of this article, we list a few of the most relevant refinements of Bertrand's postulate below. For further reading we recommend [2] and [3].

In 1958 Polish mathematician W. Sierpiński postulated that for all $n > 1$ and $k \leq n$ there exists at least one prime number in the closed interval $[kn, (k+1)\,n]$. It is obvious that the statement holds for $k = 1$ as a direct consequence of Bertrand's postulate. Recently, in 2008, M. El Bachraoui gave a proof for $k = 2$ [4]. A simple corollary of this result is that for all $n \geq 1$ the interval $\left(n, \frac{3(n+1)}{2}\right)$ contains at least one prime number. The formal proof of this refinement of Bertrand's postulate can be found in [5]. The case $k = 3$ was proved by A. Loo in 2011 [6] and it leads to a further refinement that guarantees the existence of a prime number in the interval $\left(n, \frac{4(n+2)}{3}\right)$ for all $n \geq 3$.

From the theoretical perspective, both of the refinements above are a consequence of a refinement proved by J. Nagura in 1952. Namely, he proved that for $n \geq 25$ the interval $\left(n, \frac{6n}{5}\right)$ contains at least one prime number [7]. However, his proof relies on more advanced results and concepts from Number Theory and Calculus.

Using the prime number theorem it can also be proved that for any $\epsilon > 0$ there exists n_0 such that for all $n > n_0$ the interval $(n, (1+\epsilon)\,n)$ contains at least one prime number. Note that this generalized statement does not give a precise value of n_0 and might be unsuitable for finding all solutions of a given equation or solving some similar types of problems.

4 Solved Problems

Our primary goal in this article is to demonstrate the applications of the concept of intervals containing at least one prime number and in order to achieve this goal we use Bertrand's postulate. Our key motivation for this choice is the fact that Bertrand's postulate is probably the best known among this group of theorems. On the other hand, as we show here, it is still powerful enough in solving many problems. However, we strongly encourage the reader to simplify the given solutions and strengthen some of the problem statements from this and the next section using refinements of Bertrand's postulate mentioned above.

Problem 1 *Prove that for any positive integer k there exist at least three different prime numbers having exactly k digits.*

Solution. It is trivial to verify that the statement holds for $k = 1$. For $k > 1$, we consider intervals $\left(10^{k-1}, 2 \cdot 10^{k-1}\right)$, $\left(2 \cdot 10^{k-1}, 4 \cdot 10^{k-1}\right)$ and $\left(4 \cdot 10^{k-1}, 8 \cdot 10^{k-1}\right)$. These three intervals are obviously pairwise disjoint and consist only of positive integers having exactly k digits. By Bertrand's postulate each of them contains at leat one prime number, hence the conclusion follows.

Note that $k = 1$ was considered separately since in this case $10^{k-1} = 1$ so Bertrand's postulate can not be applied to the interval $\left(10^{k-1}, 2 \cdot 10^{k-1}\right)$. $\square$

Problem 2 *Let p_n denote the n-th prime number. Prove that*

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n > p_{n+1}^2$$

holds for all $n \geq 4$. (Bonse's inequality)

Solution. We give a proof based on mathematical induction. For $n = 4$ we have

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 = 2 \cdot 3 \cdot 5 \cdot 7 = 210 > 11^2 = p_5^2$$

so the inequality holds in this case.

Assume now that it holds for all positive integers $n \leq m$. We prove that it also holds for $n = m + 1$. In this case our aim is to prove that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot p_{m+1} > p_{m+2}^2.$$

By Bertrand's postulate there exists a prime number q such that $p_{m+1} < q < 2p_{m+1}$ implying that $p_{m+2} \leq q < 2p_{m+1}$ so in order to complete our proof it is enough to show that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m \cdot p_{m+1} > 4p_{m+1}^2.$$

Obviously $p_{m+1} > 4$ so it suffices to prove that

$$p_1 \cdot p_2 \cdot \ldots \cdot p_m > p_{m+1}^2.$$

But the last inequality directly follows from the inductive assumption and this completes our proof for the case $n = m + 1$. The induction principle now implies that the given inequality holds for all $n \geq 4$. $\square$

Problem 3 *If m and n are positive integers prove that*

$$\frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n}$$

is not an integer.

Solution. Let

$$A = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{m+n}.$$

First, observe that for $n \leq m-1$ we obviously have

$$A < m \cdot \frac{1}{m} = 1$$

hence $0 < A < 1$ implying that A is not an integer in this case.

Assume now that $n \geq m$. It is trivial to verify that the problem statement holds for $n = m = 1$, so we may assume that $m+n > 2$. Then $\frac{m+n}{2} > 1$ and using Bertrand's postulate we have that there exists a prime number p such that $\frac{m+n}{2} < p < m+n$. As $2p > m+n$ and $p > \frac{m+n}{2} \geq m$, the prime number p is the only number in the closed interval $[m, m+n]$ that is divisible by p.

Now, if we bring all summands in A to a common denominator we obviously get

$$A = \frac{p \cdot B + C}{m \cdot (m+1) \dots (m+n)}$$

where $B \in \mathbb{N}$ and $C = m \cdot (m+1) \dots (p-1) \cdot (p+1) \dots (m+n)$. Analyzing the last fraction, clearly C is not divisible by p hence its numerator is not divisible by p. As p is one of the factors in its denominator the whole fraction can not be an integer and this completes our proof. $\square$

Problem 4 *Prove that the interval $\left(2^n + 1, 2^{n+1} - 1\right)$, $n \geq 2$ contains an integer that can be represented as a sum of n prime numbers.*

(Mathematical Reflections)

Solution. Solving this problem is equivalent to finding a set of prime numbers $\{p_1, p_2, \dots, p_n\}$ such that $2^n + 1 < A < 2^{n+1} - 1$, where $A = p_1 + p_2 + \cdots + p_n$. The construction of the set satisfying these conditions is given below.

Define $p_1 = 3$. By Bertrand's postulate there exist prime numbers $p_2, p_3, \dots, p_n$ such that

$$\begin{gathered} 2 < p_2 < 2^2 \\ 2^2 < p_3 < 2^3 \\ \vdots \\ 2^{n-1} < p_n < 2^n. \end{gathered}$$

We prove that $A = p_1 + p_2 + \cdots + p_n$ belongs to the interval $\left(2^n + 1, 2^{n+1} - 1\right)$.

In order to show that $A > 2^n + 1$, note that

$$A = p_1 + p_2 + \cdots + p_n > 3 + 2 + 2^2 + \cdots + 2^{n-1} =$$

$$2 + \left(1 + 2 + 2^2 + \cdots + 2^{n-1}\right) = 2 + 2^n - 1 = 2^n + 1.$$

Proving that $A < 2^{n+1} - 1$ can be done in an analogous way:

$$A = p_1 + p_2 + \cdots + p_n < 3 + 2^1 + 2^2 + \cdots + 2^n = 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1.$$

$\square$

Problem 5 *Find all positive integers m such that*

$$1! \cdot 3! \cdot 5! \cdot \dots \cdot (2m-1)! = \left(\frac{m\,(m+1)}{2}\right)!.$$

(Mediterranean M.C. 2004)

Solution. Our main idea in solving this problem is finding a prime number p which divides the right hand side (RHS), but does not divide the left hand side (LHS) of the given equation. In order to find such a prime, observe that for $m > 1$ Bertrand's postulate guarantees the existence of a prime number p such that

$$2m-1 < p < 2\,(2m-1)\,.$$

Clearly $p \nmid 1! \cdot 3! \cdot 5! \cdot \dots \cdot (2m-1)!$ and $p \leq 4m-3$.

Now, if $4m-3 \leq \frac{m(m+1)}{2}$, which is equivalent to $m \geq 6$, we have $p \leq \frac{m(m+1)}{2}$. This implies that $p \mid \left(\frac{m(m+1)}{2}\right)!$ so in this case p divides RHS, but does not divide LHS. Consequently, there is no solution for $m \geq 6$.

Therefore it remains to check cases where $m < 6$. Direct validation shows that $m = 1$ is the only solution. □

Problem 6 *For an integer $n > 3$ define $n?$ as a product of all prime numbers less than n. Find all integers $n > 3$ such that*

$$n? = 2n + 16.$$

(Russia, 2007)

Solution. The LHS of given equation is divisible by 2, but not by 4. This implies that n is an odd number (otherwise the RHS is divisible by 4). Let $n = 2k+1$ for some integer $k \geq 2$. Then our equation becomes

$$(2k+1)? = 2\,(2k+9)\,.$$

By Bertrand's postulate there exists a prime number p such that $k < p < 2k$. Then $p \mid (2k+1)?$ which implies $p \mid 2\,(2k+9)$. Since $p \geq k+1 \geq 3$, p must be an odd prime so $p \mid (2k+9)$. Now, note that $p < 2k$ implies $p < 2k+9$ so p is a proper divisor of $2k+9$. On the other hand, since $2k+9$ is odd, it is not divisble by 2 so its smallest divisor is greater than or equal to 3. As a consequence, its greatest proper divisor is not greater than $\frac{2k+9}{3}$. Since p is one of its proper divisors we conclude that $p \leq \frac{2k+9}{3}$. This leads us to the following inequality

$$k+1 \leq p \leq \frac{2k+9}{3}$$

that implies $k \leq 6$. This is equivalent to $n \leq 13$ so in order to complete our solution it remains to inspect for the values of n such that $4 \leq n \leq 13$ and $n? = 2n+16$. For $8 \leq n \leq 13$ we have $n? \geq 2 \cdot 3 \cdot 5 \cdot 7 > 2 \cdot 13 + 16 \geq 2n + 16$. Therefore we do not have solutions for $n \geq 8$. Direct verification for $n = 4, 5, 6, 7$ shows that $n = 7$ is the only solution of the given equation. □

Problem 7 *Find all positive integers n for which the number of all positive divisors of the number $\operatorname{lcm}(1,2,\ldots,n)$ is equal to 2^k for some non-negative integer k.*

(Estonia, IMO TST 2004)

Solution. Assume that we are given a fixed n. Our main idea in solving this problem is finding a prime number p such that $p^2 < n < p^3$. Namely, if such prime p exists then $\operatorname{lcm}(1,2,\ldots,n) = p^2 \cdot A$, where A is some positive integer and $\gcd(A,p) = 1$. This directly implies that the number of divisors of $\operatorname{lcm}(1,2,\ldots,n)$ is divisible by 3 and therefore it can not be equal to 2^k for some non-negative integer k.

In order to find such a prime p, observe that for $n > 4$ by Bertrand's postulate there exists a prime number p such that $\frac{\sqrt{n}}{2} < p < \sqrt{n}$. This directly implies $p^2 < n$ and it remains to discuss whether $n < p^3$ holds. Note that, by the choice of p, we have

$$p^3 > \left(\frac{\sqrt{n}}{2}\right)^3 = n \cdot \frac{\sqrt{n}}{8}.$$

It is now obvious that $p^3 > n$ holds for all $n \geq 64$ implying that there is no solution for any such n. We may now assume that in the rest of our solution $n < 64$.

Although the brute force approach is already applicable at this point, we can avoid it for most of the cases by providing an exact value of p depending on n. It is enough to observe that $p = 5$ works for all n such that $25 < n < 64$. Similarly, if $9 < n \leq 25$ we have $3^2 < n < 3^3$ leading to the choice $p = 3$. For $n \leq 9$ direct verification shows that $n = 1, 2, 3, 8$ are the only solutions. □

5 Problems for Self-study

Problem 1 *Let p_n denote the n-th prime number (i.e. $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $\ldots$). Prove that $p_n \leq 2^n$.*

Problem 2 *Find all integers $n > 1$ and $m > 1$ such that*

$$1! \cdot 3! \cdot 5! \cdots (2n-1)! = m!.$$

(American Mathematical Monthly)

Problem 3 *Determine all triplets of positive integers (k, m, n) such that*

$$n! = m^k.$$

(Singapore IMO training)

Problem 4 *Find the greatest integer that cannot be written as a sum of distinct prime numbers.*

(Mathematical Reflections)

Problem 5 *Prove that the number*

$$a_n = \sum_{k=n}^{2n} \frac{(2k+1)^n}{k}$$

is not an integer for any positive integer n.

(Austria, Regional Competition, 2008)

Problem 6 *Prove that for each positive integer k the integers $\{1, 2, \ldots, 2k\}$ can be arranged into k disjoint pairs so that the sums of the elements in each pair is prime.*

Comment: *For a proof and some interesting applications of this result see [8].*

Problem 7 *Determine all triplets (a, b, c) of positive integers that satisfy*

$$a! + b^b = c!.$$

(Crux Mathematicorum)

Problem 8 *Let n be a natural number such that m divides n for each positive integer $m < \sqrt{n}$. Prove that $n < 49$.*

(Albania NMO 2005)

Hint: We suggest using Nagura's refinement of Bertrand's postulate for the remaining three problems given below.

Problem 9 *Find the smallest positive integer n_0 such that for all integers $n > n_0$ the interval $(n, 2n)$ contains at least three prime numbers.*

Problem 10 *Let p_n denote the n-th prime number. Prove that*

$$\frac{5}{2} \geq \frac{p_{n-1} + p_{n+1}}{p_n} \geq \frac{3}{2}.$$

(www.artofproblemsolving.com)

Problem 11 *For each positive integer n , determine the least integer m such that*

$$\operatorname{lcm}\{1, 2, \ldots, m\} = \operatorname{lcm}\{1, 2, \ldots, n\}.$$

(American Mathematical Monthly)

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Where does the altitude touch down?

I. Melnik

When solving problems from stereometry, it is important to correctly determine the arrangement and positioning of the figures involved. Many mistakes made when solving these problems involve the altitude of a pyramid or a prism. If you incorrectly place the base of the altitude in relation to the base of the given pyramid or prism, as a result you will incorrectly construct the angle between the base and the lateral edge and incorrectly determine its measure, among other mistakes.

The goal of this article is to teach the reader to solve stereometry problems while avoiding aforementioned mistakes. As such, we begin the article by stating three problems with the list of potential solutions. Of course, you should not try to guess the correct answer! By analyzing these three problems, you will then easily understand the solutions of the examples that will follow.

Problem 1. Given a pyramid $SABC$, where on the plane containing the base ABC does the pyramid's altitude touch down if

a) lateral edges are congruent;

b) lateral edges form equal angles with the base of the pyramid;

c) lateral faces form equal angles with the base of the pyramid;

d) vertex angles at the apex S are all right angles;

e) two pairs of opposite edges are mutually perpendicular;

f) lateral edges are congruent and $\angle ASB = \angle ASC = 60°$, $\angle BSC = 90°$;

g) $\angle ACB = 90°$ and $AC \perp BS$;

h) $\overline{SO} = \frac{1}{3}(\overline{SA} + \overline{SB} + \overline{SC})$ where O is the center of the base?

Possible answers:

1. the point of intersection of the medians of ABC;
2. the point of intersection of the altitudes of ABC;
3. the center of the circumcircle of ABC;
4. the center of the incircle of ABC;
5. the center of the incircle of ABC or the center of one of the excircles of ABC;
6. the midpoint of BC;
7. a point on the line containing BC.

Often to correctly solve the problem, you have to determine whether the foot of the pyramid's altitude falls on the inside or the outside of the pyramid.

Problem 2. In a pyramid $SABC$, lateral faces form equal angles with the base of the pyramid. Which of the following conditions is sufficient to guarantee that the foot of the pyramid's altitude is inside the triangle base ABC?

Possible answers:

1. $\triangle ABC$ is isosceles;
2. $\triangle ABC$ is equilateral;
3. $\triangle ABC$ is a right-angle triangle or is an acute-angle triangle;
4. $\triangle ABC$ is an obtuse-angled triangle;
5. all of the above.

Problem 3. The base of the pyramid $SABCD$ is a trapezoid $ABCD$ with $BC \parallel AD$ and $|BC| < |AD|$. The lateral faces of the pyramid are congruent. Which of the following conditions is sufficient to guarantee that the foot of the pyramid's altitude is outside of the trapezoidal base $ABCD$?

Possible answers:

1. $\angle ABD \leq 90°$;
2. $\angle ABD > 90°$
3. the angle between $\overline{BD}$ and $\overline{CA}$ is greater than or equal to $90°$;
4. the angle between $\overline{BA}$ and $\overline{CD}$ is greater than $90°$;
5. none of the above.

Let us now consider two examples.

Example 1. The lateral edges of a triangular pyramid all have length l and two of the vertex angles at the apex of the pyramid are equal. Find the volume of the pyramid.

Solution. Let $SABC$ be the given pyramid with $\angle ASC = \angle BSC$ and $\angle ASB = \beta$. We immediately have that $\triangle ASC \cong \triangle BSC$ and hence $|AC| = |BC|$. Therefore, the foot O of the pyramid's altitude SO coincides with the center of the circumcircle of ABC (Problem 1a). Then we have three possibilities (see Problem 2):

1. $\angle ACB < 90°$ and the foot of the altitude is inside $\triangle ABC$ (Figure 1a);
2. $\angle ACB = 90°$ and the foot of the altitude is midpoint D of AB (Figure 1b);
3. $\angle ACB > 90°$ and the foot of the altitude is outside $\triangle ABC$ (Figure 1c).

Case a). We have:

$$|AB| = 2l \sin\frac{\beta}{2}, \quad |SD| = l\cos\frac{\beta}{2}, \quad |AC| = |BC| = 2l\sin\frac{\alpha}{2},$$

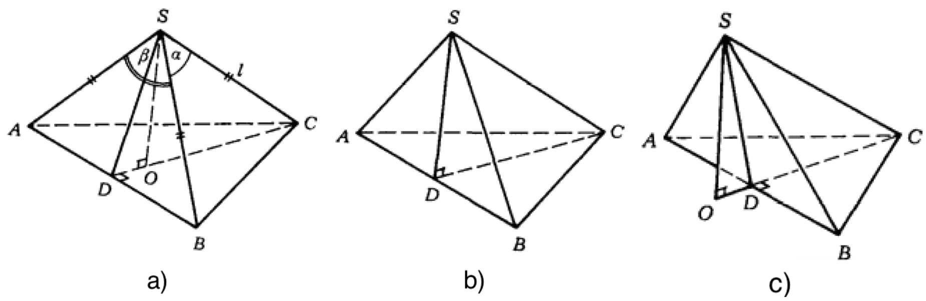


Figure 1: Possible positionings for the pyramid's altitude in Example 1.

$$|CD| = \sqrt{4l^2 \sin^2 \frac{\alpha}{2} - l^2 \sin^2 \frac{\beta}{2}} = l\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}.$$

Let $|OD| = x$. From the triangle SOD, we have

$$|SO|^2 = l^2 \cos^2 \frac{\beta}{2} - x^2.$$

From triangle CSO, we have

$$\begin{aligned} |SO|^2 &= l^2 - (|CD| - x)^2 \\ &= l^2 - l^2 + 2l^2 \cos\alpha - l^2 \cos^2 \frac{\beta}{2} - x^2 + 2xl\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}. \end{aligned}$$

Combining the above two equations and doing some manipulations, we get

$$\begin{aligned} x &= l\frac{\cos^2 \frac{\beta}{2} - \cos\alpha}{\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}}, \\ h &= l\frac{\sqrt{\cos^2 \frac{\beta}{2} - \cos^2\alpha}}{\sqrt{1 - 2\cos\alpha + \cos^2 \frac{\beta}{2}}}. \end{aligned}$$

Therefore,

$$V = \frac{1}{3}\mathcal{S}_{\triangle ABC} \cdot h = \frac{1}{3}l^3 \sin\frac{\beta}{2}\sqrt{\cos^2 \frac{\beta}{2} - \cos^2\alpha}. \tag{1}$$

Note that since $\angle ASB < \angle ASC + \angle BSC$, we have that $\beta < 2\alpha$ and hence the expression under the square root is positive.

Case b). This case is possible if and only if

$$\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}. \tag{2}$$

Indeed, if $\angle C = 90^\circ$, we have $|CD| = |AD| = l\sin\frac{\beta}{2}$ and $|CD| = |AC|\frac{\sqrt{2}}{2} = \sqrt{2}l\sin\frac{\alpha}{2}$ (see Figure 1b). Then $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$. Conversely, if $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$, then

$$|CD| = l\sqrt{1 - 2\cos\alpha + \cos^2\frac{\alpha}{2}} = \sqrt{2}l\sqrt{1 - \cos\alpha - \sin^2\frac{\alpha}{2}} = \sqrt{2}l\sin\frac{\alpha}{2}.$$

Therefore, $\cos\angle ACD = \frac{|CD|}{|AC|} = \frac{\sqrt{2}}{2}$. Hence, $\angle ACD = 45^\circ$ and $\angle C = 90^\circ$. If $\sin\frac{\beta}{2} = \sqrt{2}\sin\frac{\alpha}{2}$, then $h = |SD| = l\cos\frac{\beta}{2}$ (Figure 1b) and

$$V = \frac{l^2}{3}\sin^2\frac{\beta}{2} \times l\cos\frac{\beta}{2} = \frac{l^3}{6}\sin\beta\sin\frac{\beta}{2}.$$

Case c). Let $x = |OD|$ (Figure 1c). We then find the values for x, h and V as in Case a).

We can see that the answers in Cases a) and c) are the same even though they arose from a different situation. Moreover, these answers also fit Case b), which is not hard to see: using (2), eliminate α in (1) to get (3). Therefore, in all cases $V = \frac{1}{3}l^3\sin\frac{\beta}{2}\sqrt{\cos^2\frac{\beta}{2} - \cos^2\alpha}$. (Let us emphasize, however, that the solution which *only* considers Case a) is not complete.) $\square$

Example 2. Suppose the base of the pyramid $SABC$ is a triangle ABC with $|AB| = |BC| = 20$ and $|AC| = 32$. Lateral faces form equal angles of 45° with the base of the pyramid. Find the volume of the pyramid.

Solution. The foot of the pyramid's altitude is the point equidistant from the lines AB, AC and BC, that is, the center of the incircle of ABC or the center of one of the excircles of ABC (Problem 1c). Therefore, there are four pyramids which satisfy the conditions of the problem with points O_1, O_2, O_3 and O_4 as feet of the pyramid's altitudes (see Figure 2).

The area of the base is known to be $\mathcal{S}_{\triangle ABC} = 192$, so it remains to determine the lengths of the altitudes SO_1, SO_2, SO_3 and SO_4.

Case 1. Consider the pyramid with altitude SO_1. We have

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(20 + 20 + 32)r_1 = 36r_1 \quad\Longleftrightarrow\quad 192 = 36r_1 \quad\Longrightarrow\quad r_1 = h_1 = 5\frac{1}{3}.$$

Hence, $V_1 = 341\frac{1}{3}$.

Case 2. Pyramids with altitudes SO_2 and SO_3 are congruent. We have:

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(|AC|r_2 - |BC|r_2) = 16r_2 \quad\Longleftrightarrow\quad 192 = 16r_2 \quad\Longrightarrow\quad r_2 = h_2 = 12.$$

Hence, $V_2 = V_3 = 768$.

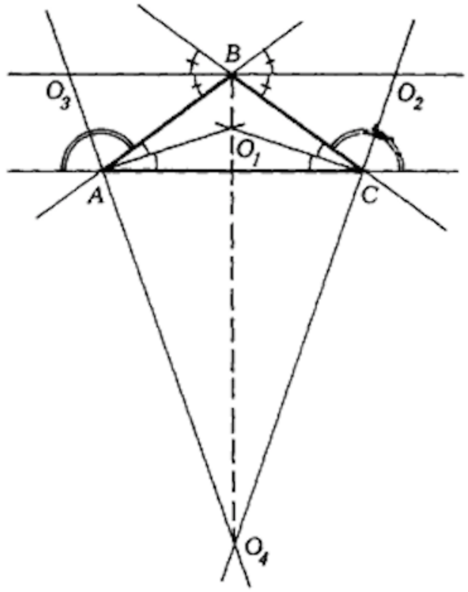


Figure 2: Possible positionings for the pyramid's altitude in Example 2.

Case 3. Finally, consider the pyramid with altitude SO_4. We have

$$\mathcal{S}_{\triangle ABC} = \frac{1}{2}(|AB|r_4 + |BC|r_4 - |AC|r_4) = 4r_4 \iff 192 = 4r_4 \implies r_4 = h_4 = 48.$$

Hence, $V_4 = 3072$.

Therefore, the set of solutions is $\{341\frac{1}{3}, 768, 3072\}$.

Exercises.

1. The base of a pyramid is an isosceles trapezoid whose parallel sides are equal to a and $2a$. Lateral faces form equal angles with the base of the pyramid and pyramid's altitude equals a. Find the lateral surface area of the pyramid.

2. The base of a pyramid $SABC$ is a triangle ABC with $|AB| = a$ and $|BC| = b$. The lateral face which goes through the edge AC is perpendicular to the base of the pyramid. The other two lateral faces form equal angles with the base of the pyramid. Find the ratio of the volumes of pyramids $SABC$ and $SOBC$, where O is the foot of the altitude of the pyramid itself.

3. The base of a pyramid $SABC$ is a right-angle triangle ABC with hypotenuse c and acute angle α. The lateral edge SC forms an angle β with the base of the pyramid. The vectors $\overline{AO}$, $\overline{BO}$ and $\overline{CO}$ (where O is the foot of the altitude of the pyramid itself) connected end to end form a triangle. Find the volume of the pyramid.

4. Consider a parallelepiped with square base and top of side length b and rhombuses for all lateral faces. One of the vertices of the top face is equidistant from all the vertices of the bottom face. Find the volume of the parallelepiped.

5. A triangular prism has an isosceles right angle triangle ABC as its base with the hypotenuse BC and $|AB| = |AC| = a$. Lateral edges AA_1, BB_1 and CC_1 form an angle of 60° with the base of the prism. The diagonal BC_1 of the lateral face CBB_1C_1 is perpendicular to the edge AC. Finally, the length of the diagonal BC_1 is equal to $a\sqrt{6}$. Find the volume of the prism.

This article appeared in Russian in Kvant, 1980(4), p. 40–43. It has been translated and adapted with permission.

Math Quotes

The mathematician who pursues his studies without clear views of this matter, must often have the uncomfortable feeling that his paper and pencil surpass him in intelligence.

Ernst Mach in "The Economy of Science" in J. R. Newman (ed.) "The World of Mathematics", New York: Simon and Schuster, 1956.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2017***; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction remercie André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

4131. *Proposé par Michel Bataille.*

Soit a, b et c des réels tels que $\tanh a \tanh b + \tanh b \tanh c + \tanh c \tanh a = 1$. Démontrer que l'équation

$$\sinh(2a - x) + \sinh(2b - x) + \sinh(2c - x) + \sinh x = 4 \sinh a \sinh b \sinh c$$

admet exactement une solution réelle.

4132. *Proposé par Marian Maciocha.*

Soit a et b des entiers tels que $a^2 + b^2$ soit un diviseur de $2a^3 + b^2$. Démontrer que l'entier $2a^3b^2 + ab^2 + 3b^4$ est divisible par l'entier $a^2 + b^2$.

4133. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit la suite (a_n) définie de façon récursive par $a_1 = 1$ et $a_{n+1} = (2n+1)!!a_n$ pour tous entiers n positifs. Calculer

$$\lim_{n\to\infty} \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}.$$

4134. *Proposé par Leonard Giugiuc, Daniel Sitaru et Qing Song.*

Soit a, b et c des réels tels que $a^2 + b^2 + c^2 = 6$ et $abc = -2$. Démontrer que

$$a + b + c \geq 0 \quad \text{ou} \quad a + b + c \leq -4.$$

4135. *Proposé par Daniel Sitaru.*

Soit un triangle ABC avec $BC = a$, $AC = b$ et $AB = c$. Démontrer que

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{3\Big(\frac{a^2}{b+c-a} + \frac{b^2}{a+c-b} + \frac{c^2}{a+b-c}\Big)}.$$

4136. *Proposé par Daniel Sitaru et Mihaly Bencze.*

Soit $a, b, c \in (0, \infty)$. Démontrer que

$$b \int_0^a e^{-t^2} dt + c \int_0^b e^{-t^2} dt + a \int_0^c e^{-t^2} < \frac{\pi}{2} \sqrt{3(a^2+b^2+c^2)}.$$

4137. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit n un entier ($n \geq 4$) et soit a, b et c trois vecteurs réels unitaires à n dimensions, orthogonaux deux à deux. Démontrer que $a_i^2 + b_i^2 + c_i^2 \leq 1$ pour tous i.

4138. *Proposé par Lorian Saceanu.*

a) Soit ABC un triangle acutangle ayant pour demi-périmètre p. Soit r le rayon du cercle inscrit dans le triangle et R le rayon du cercle circonscrit au triangle. Démontrer que

$$a \sin \frac{A}{2} + b \sin \frac{B}{2} + c \sin \frac{C}{2} \geq p + \frac{p(R-2r)^2}{4R(R+2r)}.$$

b) Soit ABC un triangle scalène ayant pour demi-périmètre p. Soit r le rayon du cercle inscrit dans le triangle et R le rayon du cercle circonscrit au triangle. Démontrer que

$$a \sin \frac{A}{2} + b \sin \frac{B}{2} + c \sin \frac{C}{2} \geq p + \frac{p^2 - 12Rr - 3r^2}{2\sqrt{6R(4R+r)}}.$$

4139. *Proposé par Ángel Plaza.*

Soit $f : [-1, 1] \to \mathbb{R}$ une fonction n fois dérivable de manière que $f^{(n)}$ soit continue, que $f(0) = 0$ et que $f^{(i)}(0) = 0$ pour tous i inférieurs ou égaux à n. Démontrer que

$$\int_{-1}^{1} (f^{(n)}(x))^2 \; dx \geq \frac{(2n+1)(n!)^2}{2} \left(\int_{-1}^{1} f(x) \; dx \right)^2.$$

4140. *Proposé par Mihaela Berindeanu.*

Déterminer tous les entiers strictement positifs x et y pour lesquels $x \leq y$ et le nombre

$$p = \frac{(x+y)(xy-4)}{xy+13}$$

est premier.

. . . .

4131. *Proposed by Michel Bataille.*

Let a, b, c be real numbers such that

$$\tanh a \tanh b + \tanh b \tanh c + \tanh c \tanh a = 1.$$

Show that the equation

$$\sinh(2a - x) + \sinh(2b - x) + \sinh(2c - x) + \sinh x = 4 \sinh a \sinh b \sinh c$$

has exactly one real solution.

4132. *Proposed by Marian Maciocha.*

Let a and b be integers such that $a^2 + b^2$ divides $2a^3 + b^2$. Prove that the integer $2a^3b^2 + ab^2 + 3b^4$ is divisible by $a^2 + b^2$.

4133. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequence (a_n) defined recursively by $a_1 = 1$ and $a_{n+1} = (2n+1)!!a_n$ for all positive integers n. Compute

$$\lim_{n\to\infty} \frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[n^2]{a_n}}.$$

4134. *Proposed by Leonard Giugiuc, Daniel Sitaru and Qing Song.*

Let a, b and c be real numbers such that $a^2 + b^2 + c^2 = 6$ and $abc = -2$. Prove that

$$a + b + c \geq 0 \quad \text{or} \quad a + b + c \leq -4.$$

4135. *Proposed by Daniel Sitaru.*

Let ABC be a triangle with $BC = a$, $AC = b$, $AB = c$. Prove that the following relationship holds

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{3\Big(\frac{a^2}{b+c-a} + \frac{b^2}{a+c-b} + \frac{c^2}{a+b-c}\Big)}.$$

4136. *Proposed by Daniel Sitaru and Mihaly Bencze.*

Prove that if $a, b, c \in (0, \infty)$ then:

$$b\int_0^a e^{-t^2}dt + c\int_0^b e^{-t^2}dt + a\int_0^c e^{-t^2} < \frac{\pi}{2}\sqrt{3(a^2+b^2+c^2)}.$$

4137. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let $n \geq 4$ be an integer and let a, b, c be three real n-dimensional vectors which are pairwise orthogonal and of unit length. Prove that $a_i^2 + b_i^2 + c_i^2 \leq 1$ for all i.

4138. *Proposed by Lorian Saceanu.*

a) Let ABC be an acute triangle with semi-perimeter s, inradius r and circumradius R. Prove that

$$a\sin\frac{A}{2} + b\sin\frac{B}{2} + c\sin\frac{C}{2} \geq s + \frac{s(R-2r)^2}{4R(R+2r)}.$$

b) Let ABC be a scalene triangle with semi-perimeter s, inradius r and circumradius R. Prove that

$$a\sin\frac{A}{2} + b\sin\frac{B}{2} + c\sin\frac{C}{2} \geq s + \frac{s^2 - 12Rr - 3r^2}{2\sqrt{6R(4R+r)}}.$$

4139. *Proposed by Ángel Plaza.*

Let $f : [-1, 1] \to \mathbb{R}$ be a n times differentiable function with $f^{(n)}$ continuous such that $f(0) = 0$, and $f^{(i)}(0) = 0$ for all even i less than or equal to n. Show that

$$\int_{-1}^{1} (f^{(n)}(x))^2 \, dx \geq \frac{(2n+1)(n!)^2}{2} \left(\int_{-1}^{1} f(x) \, dx \right)^2.$$

4140. *Proposed by Mihaela Berindeanu.*

Find all positive integers $x \leq y$ so that

$$p = \frac{(x+y)(xy-4)}{xy+13}$$

is prime.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2015: 41(4), p. 169–172.

4031. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Prove that

$$\frac{2F_1^4+F_2^4+F_3^4}{F_1^2+F_3^2}+\frac{2F_2^4+F_3^4+F_4^4}{F_2^2+F_4^2}+\cdots+\frac{2F_n^4+F_1^4+F_2^4}{F_n^2+F_2^2}>2F_nF_{n+1},$$

where F_n represents the nth Fibonacci number ($F_0=0, F_1=1$ and $F_{n+2}=F_n+F_{n+1}$ for all $n\geq 1$).

We received five correct solutions. We present two solutions.

Editor's comments. When $n=1$, the interpretation of the left side is not clear, while when $n=2$, we obtain equality. Therefore, we suppose that $n\geq 3$.

Solution 1, by Adnan Ali and the proposers (independently).

Observe that, for positive x,y,z,

$$\frac{2x^2+y^2+z^2}{x+z}\geq x+y$$

with equality if and only if $x=y=z$, since this inequality is equivalent to $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$. It follows that the left side of the inequality is greater than

$$(F_1^2+F_2^2)+(F_2^2+F_3^2)+\cdots+(F_{n-1}^2+F_n^2)+(F_n^2+F_1^2)=2\sum_{k=1}^{n}F_k^2.$$

Since $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$ (easily obtained by induction for $n\geq 1$), the result follows.

Solution 2, by Arkady Alt.

For positive x,y,z,

$$\frac{x^2}{y+z}\geq\frac{4x-y-z}{4}$$

with equality iff $2x=y+z$. This implies that

$$\begin{aligned}\frac{2F_i^4+F_j^4+F_k^4}{F_i^2+F_k^2}&>\frac{2(4F_i^2-F_i^2-F_k^2)+(4F_j^2-F_i^2-F_k^2)+(4F_k^2-F_i^2-F_k^2)}{4}\\&=F_i^2+F_j^2,\end{aligned}$$

for distinct i, j, k (since not all of F_i, F_j, F_k are equal to $F_i + F_k$). Thus the left side is greater than $2\sum_{k=1}^{n} F_k^2 = 2F_nF_{n+1}$.

4032. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Prove that in any triangle ABC with sides a, b and c, inradius r and exradii r_a, r_b, r_c, we have:

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(r_a + r_b + r_c)}.$$

We received 13 correct solutions. We present two solutions.

Solution 1, by Titu Zvonaru.

Using Ravi's substitutions ($a = y + z$, $b = z + x$, $c = x + y$, with $x, y, z > 0$), we have

$$[ABC] = \sqrt{xyz(x + y + z)},\ r = \sqrt{\frac{xyz}{x + y + z}},\ r_a = \frac{\sqrt{xyz(x + y + z)}}{x},$$

so that

$$r(r_a + r_b + r_c) = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = xy + yz + zx.$$

We have to prove that

$$\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)} \geq 2\sqrt{3(xy + yz + zx)}.$$

Using Minkowski's inequality and the inequality $(x + y + z)^2 \geq 3(xy + yz + zx)$, we obtain

$$\begin{aligned}
&\sqrt{(x + y)(y + z)} + \sqrt{(y + z)(z + x)} + \sqrt{(z + x)(x + y)} \\
&= \sqrt{x^2 + (xy + yz + zx)} + \sqrt{y^2 + (xy + yz + zx)} + \sqrt{z^2 + (xy + yz + zx)} \\
&\geq \sqrt{(x + y + z)^2 + (3\sqrt{xy + yz + zx})^2} \\
&\geq \sqrt{3(xy + yz + zx) + 9(xy + yz + zx)} \\
&= 2\sqrt{3(xy + yz + zx)}.
\end{aligned}$$

Equality holds if and only if $x = y = z$; that is, if and only if triangle ABC is equilateral.

Solution 2, composite of similar solutions by Sĕfket Arslanagic and Kee-Wai Lau.

By the known equality

$$r_a + r_b + r_c = 4R + r,$$

the given inequality is equivalent to

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 2\sqrt{3r(4R + r)}.$$

By the AM-GM inequality,

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \geq 3 \cdot \sqrt[3]{abc}.$$

It therefore suffices to show that

$$3 \cdot \sqrt[3]{abc} \geq 2 \cdot \sqrt{3r(4R + r)},$$

which is successively equivalent to

$$\begin{aligned} 3 \cdot \sqrt[3]{4Rrs} &\geq 2 \cdot \sqrt{3r(4R+r)} \\ 3^6 \cdot 16R^2r^2s^2 &\geq 2^6 \cdot 27r^3(4R+r)^3 \\ 27R^2s^2 &\geq 4r(4R+r)^3. \end{aligned}$$

By the inequalities $s^2 \geq 16Rr - 5r^2$ due to J.C. Gerretsen and $R \geq 2r$ due to L. Euler, we have

$$\begin{aligned} 27R^2s^2 - 4r(4R+r)^3 &\geq 27(16Rr - 5r^2)R^2 - 4r(4R+r)^3 \\ &= r(R-2r)(176R^2 + 25Rr + 2r^2) \\ &\geq 0. \end{aligned}$$

This proves the inequality of the problem. Equality holds for the equilateral triangle.

4033. *Proposed by Salem Malikic.*

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be positive real numbers and $x_1, \ldots, x_n$ be real numbers such that $x_1 + \cdots + x_n = 1$ and $\alpha_i x_i + \beta_i \geq 0$ for all $i = 1, \ldots, n$. Find the maximum value of

$$\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n}.$$

We received eight correct solutions and one incorrect solution. We present a composite of three nearly identical solutions given independently by Adnan Ali, Joe Schlosberg, and Titu Zvonaru.

Let M denote the required maximum value. Applying the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} &\sqrt{\alpha_1 x_1 + \beta_1} + \sqrt{\alpha_2 x_2 + \beta_2} + \cdots + \sqrt{\alpha_n x_n + \beta_n} \\ &\quad = \sqrt{\alpha_1}\sqrt{x_1 + \frac{\beta_1}{\alpha_1}} + \sqrt{\alpha_2}\sqrt{x_2 + \frac{\beta_2}{\alpha_2}} + \cdots + \sqrt{\alpha_n}\sqrt{x_n + \frac{\beta_n}{\alpha_n}} \\ &\quad \geq \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(x_1 + x_2 + \cdots + x_n + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})} \\ &\quad = \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(1 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})}. \end{aligned}$$

The equality holds if and only if for some k we have

$$\frac{x_1 + \frac{\beta_1}{\alpha_1}}{\alpha_1} = \frac{x_2 + \frac{\beta_2}{\alpha_2}}{\alpha_2} = \cdots = \frac{x_n + \frac{\beta_n}{\alpha_n}}{\alpha_n} = k.$$

Since

$$k = \frac{\sum x_i + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i} = \frac{1 + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i},$$

we have

$$x_i = \alpha_i \cdot \frac{1 + \sum \frac{\beta_i}{\alpha_i}}{\sum \alpha_i} - \frac{\beta_i}{\alpha_i}, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{1}$$

Therefore,

$$M = \sqrt{(\alpha_1 + \alpha_2 + \cdots + \alpha_n)(1 + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_n}{\alpha_n})}$$

attained when x_i are as in (1).

4034. *Proposed by Michel Bataille.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{16^n} \sum_{k=0}^{2n} \frac{(-1)^k}{2n+1-k} \binom{2k}{k} \binom{4n-2k}{2n-k}.$$

We received three correct solutions. We present the solution by Ángel Plaza.

Let us denote

$$a_n = \sum_{k=0}^{n} \frac{(-1)^k}{n+1-k} \binom{2k}{k} \binom{2n-2k}{n-k},$$

so the proposed expression reads as

$$\sum_{n=0}^{\infty} a_{2n} \left(\frac{-1}{16}\right)^n.$$

We will use the snake oil method to find the generating function of the sequence with general term a_n. If $F(x)$ is its generating function then

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} a_n x^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{(-1)^k}{n+1-k}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n \\
&= \sum_{k\geq 0}(-1)^k\binom{2k}{k}\sum_{n\geq k}\frac{1}{n+1-k}\binom{2n-2k}{n-k}x^n \\
&= \sum_{k\geq 0}(-1)^k\binom{2k}{k}x^k\sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n} \\
&= \sum_{k\geq 0}\binom{2k}{k}(-x)^k\frac{2}{1+\sqrt{1-4x}} \\
&= \frac{1}{\sqrt{1+4x}}\cdot\frac{2}{1+\sqrt{1-4x}},
\end{aligned}$$

where we used the generating functions of the Catalan number and of the central binomial coefficients (both with radius of convergence $|x| < \frac{1}{4}$) in the last two steps. Now

$$\sum_{n\geq 0} a_{2n}x^n = \frac{F(\sqrt{x})+F(-\sqrt{x})}{2},$$

so the proposed sum is equal to

$$\frac{1}{\sqrt{1+i}}\cdot\frac{1}{1+\sqrt{1-i}}+\frac{1}{\sqrt{1-i}}\cdot\frac{1}{1+\sqrt{1+i}},$$

which can be calculated to $\sqrt{2}-\sqrt{\sqrt{2}-1}$.

4035. *Proposed by Daniel Sitaru and Leonard Giugiuc.*

Let a and b be two real numbers such that $ab = 225$. Find all real solutions (in real 2×2 matrices) to the matrix equation

$$X^3 - 5X^2 + 6X = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}.$$

We received four submissions for this question, of which three were correct and complete. We present the solution by Michel Bataille.

We will show that the solutions are the three matrices

$$X_1 = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

Simple calculations show that these three matrices satisfy the given equation.

Let $A = \begin{pmatrix} 15 & a \\ b & 15 \end{pmatrix}$ and let I_2 be the 2×2 unit matrix. Since $\det(xI_2 - A) = x^2 - 30x$, the eigenvalues of A are 0 and 30.

Let X be a solution of the given equation and let λ be an eigenvalue of X. Then $\lambda^3 - 5\lambda^2 + 6\lambda$ is an eigenvalue of A, hence $\lambda^3 - 5\lambda^2 + 6\lambda \in \{0, 30\}$. Thus,

$$\lambda(\lambda - 2)(\lambda - 3) = 0 \quad \text{or} \quad (\lambda - 5)(\lambda^2 + 6) = 0$$

and the possible eigenvalues of X are $0, 2, 3, 5, i\sqrt{6}, -i\sqrt{6}$.

Noting that the characteristic polynomial $\chi(x)$ of the real matrix X is in $\mathbb{R}[x]$, if $i\sqrt{6}$ (resp. $-i\sqrt{6}$) is an eigenvalue of X, so is its complex conjugate and then X is similar to $\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix}$. But then $A = X^3 - 5X^2 + 6X$ is similar to

$$\begin{pmatrix} -6i\sqrt{6} & 0 \\ 0 & 6i\sqrt{6} \end{pmatrix} - 5\begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix} + 6\begin{pmatrix} i\sqrt{6} & 0 \\ 0 & -i\sqrt{6} \end{pmatrix} = \begin{pmatrix} 30 & 0 \\ 0 & 30 \end{pmatrix},$$

a contradiction since 0 is an eigenvalue of A.

As a result, if λ_1, λ_2 are the eigenvalues of X (possibly $\lambda_1 = \lambda_2$), we have $\lambda_1, \lambda_2 \in \{0, 2, 3, 5\}$. Since the eigenvalues of X are real, X is similar to an upper triangular real matrix, say,

$$X = P\begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$$

for some invertible real matrix P and some real number α. Then

$$A = X^3 - 5X^2 + 6X = P\begin{pmatrix} \lambda_1^3 - 5\lambda_1^2 + 6\lambda_1 & \beta \\ 0 & \lambda_2^3 - 5\lambda_2^2 + 6\lambda_2 \end{pmatrix} P^{-1}$$

where $\beta \in \mathbb{R}$. Taking traces, it follows that

$$30 = (\lambda_1^3 - 5\lambda_1^2 + 6\lambda_1) + (\lambda_2^3 - 5\lambda_2^2 + 6\lambda_2) \qquad (1).$$

After trying all possibilities for λ_1 and λ_2, we realize that $\lambda_1 \neq \lambda_2$ and $\{\lambda_1, \lambda_2\}$ is either $\{0, 5\}$ or $\{2, 5\}$ or $\{3, 5\}$. We consider the three cases in turn:

- if $\{\lambda_1, \lambda_2\} = \{0, 5\}$, then $\chi(x) = x^2 - 5x$. Note that

  $$x^3 - 5x^2 + 6x = x(x^2 - 5x) + 6x = x\chi(x) + 6x.$$

  By the Cayley-Hamilton theorem, $\chi(X) = 0$; replacing x by X in the above we get $A = X\chi(X) + 6X = 6X$, hence $X = \frac{1}{6}A = \begin{pmatrix} 5/2 & a/6 \\ b/6 & 5/2 \end{pmatrix}$.

- if $\{\lambda_1, \lambda_2\} = \{2, 5\}$, then $\chi(x) = (x - 2)(x - 5)$ and

  $$x^3 - 5x^2 + 6x = (x + 2)\chi(x) + 10x - 20.$$

  Therefore $A = 10X - 20I_2$ and

  $$X = \frac{1}{10}(A + 20I_2) = \begin{pmatrix} 7/2 & a/10 \\ b/10 & 7/2 \end{pmatrix}.$$

- if $\{\lambda_1, \lambda_2\} = \{3, 5\}$, then $\chi(x) = (x-3)(x-5)$ and

$$x^3 - 5x^2 + 6x = (x+3)\chi(x) + 15x - 45.$$

Hence $A = 15X - 45$, which gives us

$$X = \frac{1}{15}(A + 45I_2) = \begin{pmatrix} 4 & a/15 \\ b/15 & 4 \end{pmatrix}.$$

The proof is complete.

4036. *Proposed by Arkady Alt.*

Let a, b and c be non-negative real numbers. Prove that for any real $k \geq \frac{11}{24}$ we have:

$$k(ab + bc + ca)(a + b + c) - (a^2c + b^2a + c^2b) \leq \frac{(3k-1)(a+b+c)^3}{9}.$$

We received five submissions all of which are correct. We present the solution by the proposer, slightly modified by the editor.

Due to cyclic symmetry of the functions involved, we may assume that $c = \min\{a, b, c\}$.

Let $x = a - c$ and $y = b - c$. Then $x, y, c \geq 0, a = x + c,\ b = y + c$, and $a + b + c = x + y + 3c$.

The given inequality is equivalent to

$$(3k-1)(a+b+c)^3 - 9k(ab+bc+ca)(a+b+c) + 9\left(a^2c + b^2a + c^2b\right) \geq 0$$

or

$$\begin{aligned}(3k-1)(x+y+3c)^3 - 9k((x+c)(y+c) + c(x+y+2c))(x+y+3c)\\ + 9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c)) \geq 0. \quad (1)\end{aligned}$$

Let $F(x, y, c)$ denote the expression on the left hand side of (1), and set $p = x + y$ and $q = xy$.
Since $9k((x+c)(y+c)+c(x+y+2c))(x+y+3c) = 9k(3c^2+2pc+q)(p+3c)$ and

$$\begin{aligned}&9((x+c)^2c + (y+c)^2(x+c) + c^2(y+c))\\ &= 9(cx^2 + cy^2 + 2cxy + 3c^2(x+y) + xy^2 + 3c^3)\\ &= 9(cp^2 + 3c^2p + 3c^3 + xy^2)\\ &= 9cp^2 + 27c^2p + 27c^3 + 9xy^2\\ &= (p+3c)^3 - p^3 + 9xy^2,\end{aligned}$$

we have

$$\begin{aligned}
F(x,y,c) &= (3k-1)(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) + (p+3c)^3 - p^3 + 9xy^2 \\
&= 3k(p+3c)^3 - 9k(3c^2+2pc+q)(p+3c) - p^3 + 9xy^2 \\
&= 3k(p^3+9cp^2+27c^2p+27c^3) - 9k(2cp^2+9c^2p+9c^3+pq+3cq) \\
&\qquad\qquad - p^3 + qxy^2 \\
&= (3k-1)p^3 + 9ckp^2 - 27ckq - 9kpq + 9xy^2 \\
&= (3k-1)(x+y)^3 + 9ck(x+y)^2 - 27ckxy - 9kxy(x+y) + 9xy^2 \\
&= (3k-1)(x^3+y^3+3xy(x+y)) + 9ck(x^2+2xy+y^2) - 27ckxy \\
&\qquad\qquad - 9kxy(x+y) + 9xy^2 \\
&= (3k-1)x^3 + 6xy^2 - 3x^2y + 9ck(x^2-xy+y^2) + (3k-1)y^3. \qquad (2)
\end{aligned}$$

Clearly, $9ck(x^2-xy+y^2)+(3k-1)y^3 \geq 0$. Furthermore,

$$(3k-1)x^3 + 6xy^2 - 3x^2y = x((3k-1)x^2 - 3xy + 6y^2) \geq 0$$

since the discriminant of $(3k-1)x^2 - 3xy + 6y^2$ is

$$9y^2 - 24(3k-1)y^2 = 3(11-24k)y^2 \leq 0$$

and $3k-1>0$. Hence, from (2) we conclude that $F(x,y,c) \geq 0$ which by (1) completes the proof.

4037. *Proposed by Michel Bataille.*

Let P be a point of the incircle γ of a triangle ABC. The perpendiculars to BC, CA and AB through P meet γ again at U, V and W, respectively. Prove that the area of UVW is independent of the chosen point P on γ.

We received six correct and complete solutions. We present the solution by Oliver Geupel. Ricard Peiró and Prithwijt De submitted similar solutions.

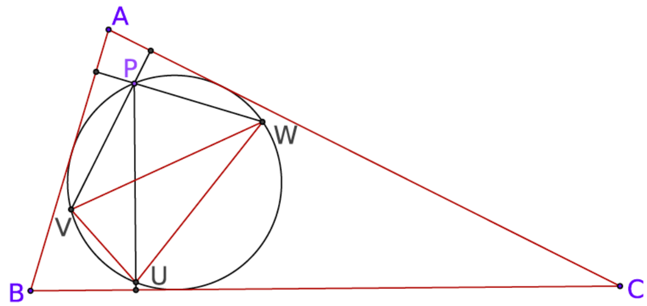
We prove that triangle UVW is similar to triangle ABC. As a consequence, since γ is the circumcircle of UVW, the area of triangle UVW is

$$[UVW] = \frac{r^2}{R^2}[ABC],$$

where r and R denote the inradius and the circumradius, respectively, of $\triangle ABC$.

Let $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{U}$, $\hat{V}$, and $\hat{W}$ denote measures of the the interior angles of the triangles ABC and UVW. Since $PV \perp AC$ and $PW \perp AB$, the size of $\angle VPW$ is $180° - \hat{A}$. Also, since the points P, U, V, and W are concyclic, $\angle VUW$ is equal to either $\angle VPW$ or $180° - \angle VPW$. Hence, $\hat{U} \in \{\hat{A},\ 180° - \hat{A}\}$. Analogously, $\hat{V} \in \{\hat{B},\ 180° - \hat{B}\}$ and $\hat{W} \in \{\hat{C},\ 180° - \hat{C}\}$.

We show that $(\hat{U}, \hat{V}, \hat{W}) = (\hat{A}, \hat{B}, \hat{C})$.



Assume the contrary. Then, there is no loss of generality in assuming that $\hat{U} \neq \hat{A}$. Thus, $\hat{U} = 180° - \hat{A}$.

If $(\hat{V}, \hat{W}) = (\hat{B}, \hat{C})$, then we obtain $180° = \hat{U} + \hat{V} + \hat{W} = 180° - \hat{A} + \hat{B} + \hat{C}$, so that $\hat{A} = \hat{B} + \hat{C} = 90° = 180° - \hat{A} = \hat{U}$, contradicting our assumption that $\hat{U} \neq \hat{A}$. Hence, $(\hat{V}, \hat{W}) \neq (\hat{B}, \hat{C})$. There is no loss of generality in assuming that $\hat{V} = 180° - \hat{B}$. But then, $\hat{U} + \hat{V} = (180° - A) + (180° - B) > 180°$, a contradiction.

Editor's Comments. We were pleased to find that among the six solutions submitted, four different formulas for the area of a triangle were used.

4038. *Proposed by George Apostolopoulos.*

Let x, y, z be positive real numbers such that $x + y + z = xyz$. Find the minimum value of the expression

$$\sqrt{\frac{1}{3}x^4 + 1} + \sqrt{\frac{1}{3}y^4 + 1} + \sqrt{\frac{1}{3}z^4 + 1}.$$

There were 21 correct solutions, with four from one submitter and three from another. An additional solution was incorrect.

Solution 1, by Arkady Alt, Šefket Arslanagić, and Daniel Dan (independently).

Since $xyz = x+y+z \geq 3\sqrt[3]{xyz}$, if follows that $x+y+z = xyz \geq 3\sqrt{3}$. Applying the inequality of the root mean square and arithmetic mean, we have, for $t = x, y, z$,

$$\sqrt{\frac{1}{3}t^4 + 1} = \sqrt{\left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + \left(\frac{t^2}{3}\right)^2 + 1}$$
$$\geq \frac{(t^2/3) + (t^2/3) + (t^2/3) + 1}{2} = \frac{t^2+1}{2}$$

with equality iff $t = \sqrt{3}$. (Alternatively, the inequality $\sqrt{\frac{1}{3}t^4 + 1} \geq \frac{t^2+1}{2}$ is equivalent to $(t^2 - 3)^2 \geq 0$.) Therefore, the left side of the inequality is not less than

$$\frac{1}{2}(3 + x^2 + y^2 + z^2) \geq \frac{1}{2}\left(3 + \frac{(x+y+z)^2}{3}\right) \geq \frac{1}{2}\left(3 + (27/3)\right) = 6.$$

Since equality occurs when $x = y = z = \sqrt{3}$, the desired minimum is 6.

Solution 2, by Šefket Arslanagić and Salem Malikić (independently).

As before, $x+y+z \geq 3\sqrt{3}$. We begin by establishing that

$$\sqrt{\frac{1}{3}u^4+1} \geq u\sqrt{3}-1,$$

with equality iff $u=\sqrt{3}$. The result is clear when $u<1/\sqrt{3}$. When $u \geq 1/\sqrt{3}$, the inequality is equivalent to

$$\frac{1}{3}u^4+2\sqrt{3}u \geq 3u^2.$$

By the arithmetic-geometric means inequality, we obtain that

$$\frac{1}{3}u^4+2\sqrt{3}u = \frac{1}{3}u^4+u\sqrt{3}+u\sqrt{3} \geq 3\sqrt[3]{\frac{1}{3}u^4 \cdot u\sqrt{3}\cdot u\sqrt{3}} = 3u^2$$

as desired.

The left side of the inequality of the problem is not less than

$$\sqrt{3}(x+y+z)-3 = 9-3 = 6,$$

with equality iff $x=y=z=\sqrt{3}$. The desired minimum is 6.

Solution 3, by Titu Zvonaru.

From the triangle inequality in Euclidean space $\mathbb{R}^2$,

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|_2 \leq \|\mathbf{a}\|_2+\|\mathbf{b}\|_2+\|\mathbf{c}\|_2,$$

applied to $\mathbf{a}=(x^2/\sqrt{3},1)$, $\mathbf{b}=(y^2/\sqrt{3},1)$, $\mathbf{c}=(z^2/\sqrt{3},1)$, we have that

$$\begin{aligned}\sqrt{\frac{x^4}{3}+1}+\sqrt{\frac{y^4}{3}+1}+\sqrt{\frac{z^4}{3}+1} &\geq \sqrt{\left(\frac{x^2}{\sqrt{3}}+\frac{y^2}{\sqrt{3}}+\frac{z^2}{\sqrt{3}}\right)^2+(1+1+1)^2}\\ &= \sqrt{\frac{(x^2+y^2+z^2)^2}{3}+9},\end{aligned}$$

with equality iff $x=y=z=\sqrt{3}$. Using $x^2+y^2+z^2 \geq xy+yz+zx$, the arithmetic-harmonic means inequality and $x+y+z=xyz$ in turn, we obtain that

$$x^2+y^2+z^2 \geq xy+yz+zx \geq \frac{9xyz}{x+y+z} = 9.$$

Thus the left side of the inequality is not less than

$$\sqrt{\frac{81}{3}+9} = 6,$$

with equality iff $x=y=z=\sqrt{3}$. The desired minimum is 6.

Solution 4, by Ali Adnan.

By the Cauchy-Schwarz inequality,

$$\sqrt{\frac{1}{3}t^4+1}\cdot\sqrt{3+1}\geq t^2+1,$$

and by the arithmetic-geometric means inequality,

$$(x^2+y^2+z^2)(x+y+z)\geq 3(xyz)^{2/3}\cdot 3(xyz)^{1/3}=9xyz.$$

Therefore

$$\begin{aligned}\sqrt{\frac{1}{3}x^4+1}+\sqrt{\frac{1}{3}y^4+1}+\sqrt{\frac{1}{3}z^4+1} &\geq \frac{1}{2}(x^2+y^2+z^2)+\frac{3}{2}\\ &\geq \frac{1}{2}\left(\frac{9xyz}{x+y+z}\right)+\frac{3}{2}=6,\end{aligned}$$

with equality if and only if $x=y=z=\sqrt{3}$.

Editor's comments. Seven solvers applied Jensen's Inequality to obtain the result, the function $\sqrt{(x^4/3)+1}$ being convex. Some solvers noted that, under the stated constraint, we can write $(x,y,z)=(\tan\alpha,\tan\beta,\tan\gamma)$ with $\alpha+\beta+\gamma=\pi$ and each angle less than $\pi/2$, or $(x,y,z)=(\cot\lambda+\cot\mu+\cot\nu)$ with $\lambda+\mu+\nu=\pi/2$, and then obtain a trigonometric inequality.

4039. *Proposed by Abdilkadir Altinaş.*

In a triangle ABC, let $\angle CAB=48^\circ$ and $\angle CBA=12^\circ$. Suppose D is a point on AB such that $CD=1$ and $AB=\sqrt{3}$. Find $\angle DCB$.

We received 17 correct solutions and will feature the solution submitted by the Skidmore College Problem Group.

Let $\angle DCB=\gamma$; we shall show that $\gamma=6^\circ$.

By the Law of Sines applied to triangle ABC, $\frac{\sin 12^\circ}{AC}=\frac{\sin 120^\circ}{\sqrt{3}}=\frac{1}{2}$, so we have

$$AC=2\sin 12^\circ.$$

For triangle ACD, $\frac{\sin(12^\circ+\gamma)}{AC}=\frac{\sin 48^\circ}{1}$, whence

$$\sin(12^\circ+\gamma)=2\sin 48^\circ\sin 12^\circ=\cos 36^\circ-\cos 60^\circ=\cos 36^\circ-\frac{1}{2}.$$

Let $\phi=\frac{1+\sqrt{5}}{2}$ $(=\phi^2-1)$ be the golden section (that is, the ratio of a diagonal of the regular pentagon to a side). We know that

$$\sin 18^\circ=\frac{1}{2\phi}=\frac{\phi}{2}-\frac{1}{2}\quad\text{and}\quad\cos 36^\circ=\frac{\phi}{2}.$$

[For example, if the regular pentagon $PQRST$ has unit sides, then the isosceles triangle PRS has apex angle 36° and sides ϕ and 1 (which provides the value for

$\sin 18^\circ$), while the isosceles triangle PQT with base angle 36° also has sides 1 and ϕ (which gives $\cos 36^\circ$).] Thus $\cos 36^\circ = \sin 18^\circ + \frac{1}{2}$, and

$$\sin(12^\circ + \gamma) = \sin 18^\circ = \sin 162^\circ.$$

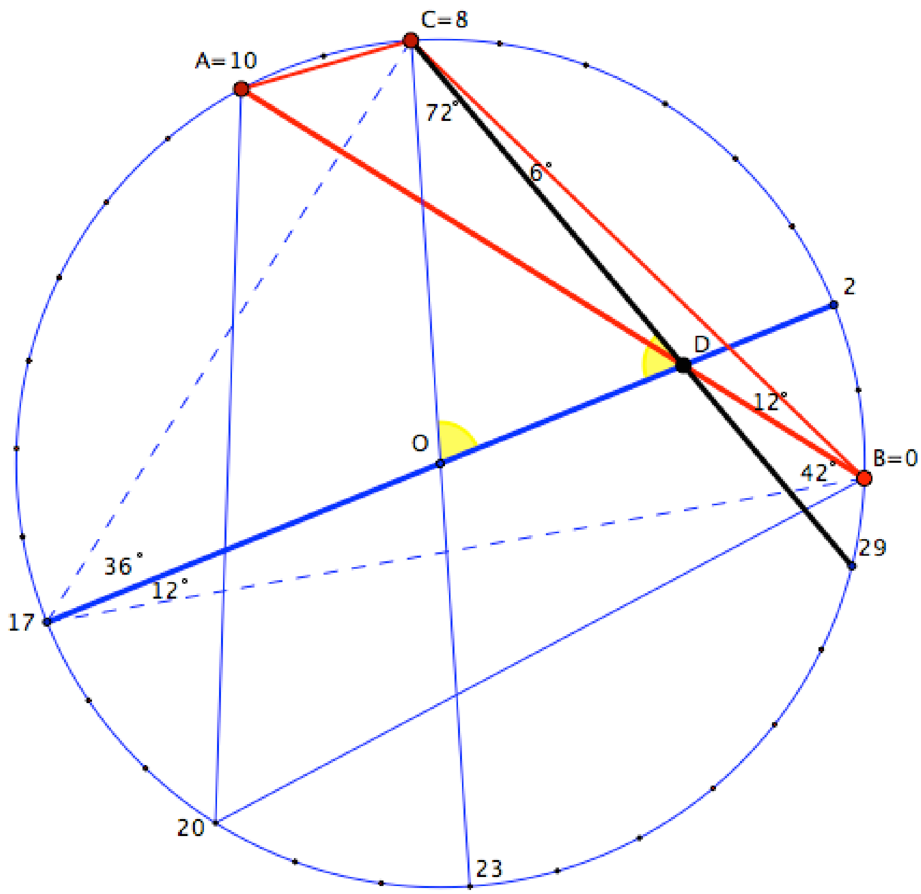
But $\gamma < 120^\circ$, which implies that $\gamma = 6^\circ$, as claimed.

Editor's comments. In a remark added to his solution, Dag Jonsson observed that there would be a second solution should we allow D to be a point of the line AB (instead of restricting it to the segment AB). Vertex A would then lie between D and B; the argument of the featured solution remains valid so that $\angle ADC = 18^\circ$, whence $\angle DCA = 30^\circ$ and, finally, $\angle DCB = 150^\circ$.

Returning to the case where D lies between A and B, we note that the degenerate quadrangle $ADBC$ of our problem is an example of an *adventitious quadrangle.* Adventitious quadrangles were first defined by Colin Tripp [5], but his narrow definition was later extended to include any quadrangle for which the angles formed by sides and diagonals are all rational multiples of 180°. Rigby [3] observed that the problem of classifying the adventitious quadrangles is equivalent to classifying all triple intersections of diagonals in regular polygons, a problem solved many years earlier by Gerrit Bol [1]. A complete account, including an elementary summary and a 15-item bibliography, was provided by Poonen and Rubinstein in [2]. Triangle ABC of our problem can be inscribed in a regular 30-gon that has a unit circumradius. Label its vertices from 0 to 29 and place A at 10, B at 0, and C at 8. Note that the angle subtended at a vertex of the 30-gon by any nonadjacent edge is 6°, which immediately implies that the angles of ΔABC are indeed $48^\circ, 12^\circ$, and 120°. Because AB is the edge of an inscribed equilateral triangle (with vertices numbered 0, 10, 20), we have $AB = \sqrt{3}$ as desired. Let D' be the point of intersection of the diameter 2, 17 and the diagonal 8, 29. If O is the circumcentre, then triangle $OD'C$ is isosceles (with angles $36^\circ, 72^\circ$, and 72°), whence $CD' = CO = 1$. It remains to prove that $D' \in AB$, which will immediately imply that $D' = D$ (and, consequently, that $\angle DCB = \angle D'CB = 6^\circ$). On p. 223 of [4] Rigby declares (with a slightly different labeling) that the diagonals 29, 8; 0, 10; and 2, 17 are indeed concurrent. While the author provides a "geometric" proof that these three lines are concurrent, it is perhaps more efficient to use trigonometry. Applying the sine form of Ceva's theorem to the triangle whose vertices are those numbered 0, 8, 17 with cevians joining 0 to 10, 8 to 29, and 17 to 2 (as in the accompanying figure), we must show that the product

$$\frac{\sin 42}{\sin 12} \cdot \frac{\sin 6}{\sin 72} \cdot \frac{\sin 36}{\sin 12}$$

equals 1. This is easily accomplished using an argument similar to that of our featured solution.



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[2] Bjorn Poonen and Michael Rubinstein, The Number of Intersection Points Made by the Diagonals of a Regular Polygon. *Siam Journal of Discrete Mathematics* **11**:1 (February 1998) 135-156.

[3] J.F. Rigby, Adventitious Quadrangles: A Geometrical Approach. *Mathematical Gazette* **62**:421 (October 1978) 183-191.

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4040. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{N} \mapsto \mathbb{N}$ such that

$$(f(a)+b)f(a+f(b)) = (a+f(b))^2 \quad \forall a,b \in \mathbb{N}.$$

There were nine submitted solutions for this problem, all of which were correct. We present three solutions; the first solution is for the case that 0 *is the first natural number, and the latter two are for the case that* 1 *is the first natural number.*

Solution 1, by Leonard Giugiuc.

Suppose that 0 is the first natural number. Let $a = b = 0$; we have:

$$f(0)f\big(f(0)\big) = f(0)^2,$$

which implies that either $f(0) = 0$ or $f\big(f(0)\big) = f(0)$. If $f\big(f(0)\big) = f(0)$, then we replace a with 0 and b with $f(0)$ in the given relation and obtain:

$$2f(0)f(0) = f(0)^2,$$

which implies that $f(0)^2 = 0$. Hence no matter what, $f(0) = 0$. Now replace b with 0 and obtain:

$$f(a)^2 = a^2,$$

for all $a \geq 0$, and so $f(a) = a$ for all $a \geq 0$, because $f(a) \geq 0$. Thus the required function is the identity (which clearly satisfies the given relation).

Solution 2, by Adnan Ali.

Suppose that 1 is the first natural number. Denote by $P(a,b)$ the above functional equation for all $a, b \in \mathbb{N}$. Now let $f(1) = k$, where k is a natural number. Then $P(1,1)$ gives $f(k+1) = k+1$. Using this, $P(1,k+1)$ implies

$$(2k+1)f(k+2) = (k+2)^2.$$

This means that $(2k+1)$ divides $(k+2)^2$. This means that $2k+1$ divides

$$4(k+2)^2 - 8(2k+1) - (2k+1)(2k-1) = 9.$$

This forces $2k+1$ to be either 3 or 9, which gives $f(1) = 1$ or $f(1) = 4$.

If $f(1) = 4$, then from $P(1,1)$ we get $f(5) = 5$. Next $P(5,1)$ gives

$$(f(5)+1)f(5+f(1)) = (5+f(1))^2 = 81,$$

implying that $6f(9) = 81$, which is clearly impossible as $6 \nmid 81$ and $f(9) \in \mathbb{N}$.

Thus we must have $f(1) = 1$. Now we prove by induction that $f(1) = 1$ implies that $f(n) = n$ for all $n \in \mathbb{N}$. The case $n = 1$ is already true. Now assume that $f(a) = a$ for some $a \geq 1$. Then from $P(a,1)$:

$$(f(a)+1)f(a+f(1)) = (a+f(1))^2,$$

which implies that $f(a+1) = a+1$, and consequently by induction $f(n) = n$ for all $n \in \mathbb{N}$. So, in summary the only function satisfying the equation is the function $f(n) = n$ for all $n \in \mathbb{N}$.

Solution 3, by Joseph Ling.

It is easy to see that $f(n) = n$ for all n satisfies the required relation for all $a, b \in \mathbb{N}$ We show that there are no other solutions.

First, we note that f is one-to-one. For if b_1 and b_2 are such that $f(b_1) = f(b_2)$, then for all a, we have

$$f(a) + b_1 = \frac{(a + f(b_1))^2}{f(a + f(b_1))} = \frac{(a + f(b_2))^2}{f(a + f(b_2))} = f(a) + b_2,$$

implying that $b_1 = b_2$.

Next, we note that f is (strictly) increasing. For if there exist $b_1 < b_2$ with $f(b_1) > f(b_2)$, then for every $a \in \mathbb{N}$, we consider $a' = a + f(b_1) - f(b_2)$. We have $a' \in \mathbb{N}$, with $a' + f(b_2) = a + f(b_1)$, and consequently,

$$f(a') + b_2 = \frac{(a' + f(b_2))^2}{f(a' + f(b_2))} = \frac{(a + f(b_1))^2}{f(a + f(b_1))} = f(a) + b_1 < f(a) + b_2$$

implying that $f(a') < f(a)$. But then this means that the range of f has no smallest element, contradicting the well-ordering principle.

Now, since f is strictly increasing, a simple induction shows that $f(n) \geq n$ for all $n \in \mathbb{N}$. It follows that $f(a + f(b)) \geq a + f(b)$ for all $a, b \in \mathbb{N}$. Consequently, the given relation implies that $f(a) + b \leq a + f(b)$ for all $a, b \in \mathbb{N}$. But then by symmetry, we conclude that in fact, $f(a) + b = a + f(b)$ for all $a, b \in \mathbb{N}$. Therefore,

$$f(n) = n + k$$

for all $n \in \mathbb{N}$, where $k = f(1) - 1$. Applying this to the relation, we get

$$(a + b + k)(a + b + 2k) = (a + b + k)^2$$

for all $a, b \in \mathbb{N}$. But then this implies that $k = 0$, and so, $f(n) = n$ for all $n \in \mathbb{N}$.

Editor's Comments. Most of the solvers, and the proposer, assumed simply that 0 was not a natural number; only Giugiuc solved the problem with both interpretations. Assuming that 0 is natural means there is more known information, and thus it is reasonable that the solution is easier than in the other case. Regarding the other case, the solution by Ali uses divisibility, a number-theoretic property, whereas the solution by Ling uses the function-theoretic properties of solutions to this relation. Other solutions use the relation differently (some use it to find additivity of the function, others show that there are infinitely many primes satisfying $f(n) = n$, etc.).

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Leonard Giugiuc, Daniel Sitaru, Romania and Qing Song, China : 4134
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Solvers - collaborations