

THE CONTEST CORNER

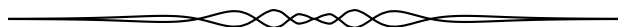
No. 44

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er janvier 2017**; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

La rédaction remercie André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.



CC216. Étant donné un ensemble de trois nombres, on crée un deuxième ensemble en remplaçant chaque nombre de l'ensemble initial par la somme des deux autres nombres de l'ensemble initial. Par exemple, à partir de l'ensemble $\{3, 4, 6\}$, on obtient l'ensemble $\{10, 9, 7\}$ et à partir de ce deuxième ensemble, on obtient l'ensemble $\{16, 17, 19\}$. En partant de l'ensemble $\{20, 1, 3\}$, on applique la procédure 2014 fois. Quelle est la plus grande différence entre n'importe quels deux nombres de l'ensemble qui en résulte?

CC217. L'hypoténuse AB d'un triangle rectangle ABC est coupée en trois parties égales par les points M et N . Sachant que $CM^2 + CN^2 = k \cdot AB^2$, quelle est la valeur de k ?

CC218. Résoudre ce système d'équations:

$$\begin{aligned} 3^{\ln x} &= 4^{\ln y} \\ (4x)^{\ln 4} &= (3y)^{\ln 3} \end{aligned}$$

CC219. Un morceau de bois, qui a la forme d'un prisme droit à base rectangulaire, mesure 4 sur 5 sur 6. On recouvre le solide de peinture verte, puis on le coupe en cubes de dimensions 1 sur 1 sur 1. Déterminer le rapport du nombre de petits cubes ayant exactement deux faces vertes au nombre de petits cubes ayant exactement trois faces vertes.

CC220. On tire au hasard deux nombres x et y de l'intervalle fermé $[0, 2]$. Quelle est la probabilité pour que $x + y > 1$?

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CC216. Starting with a list of three numbers, the “changesum” procedure creates a new list by replacing each number by the sum of the other two. For example, from $\{3, 4, 6\}$ “changesum” gives $\{10, 9, 7\}$ and a new “changesum” leads to $\{16, 17, 19\}$. If we begin with $\{20, 1, 3\}$, what is the maximum difference between two numbers of the list after 2014 consecutive “changesums”?

CC217. A right triangle ABC has its hypotenuse AB trisected at M and N . If $CM^2 + CN^2 = k \cdot AB^2$, then what is the value of k ?

CC218. Solve the following system of equations:

$$\begin{aligned}3^{\ln x} &= 4^{\ln y}, \\(4x)^{\ln 4} &= (3y)^{\ln 3}.\end{aligned}$$

CC219. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. Find the ratio of the number of cubes with exactly two green faces to the number of cubes with exactly three green faces.

CC220. Two random numbers x and y are drawn independently from the closed interval $[0, 2]$. What is the probability that $x + y > 1$?



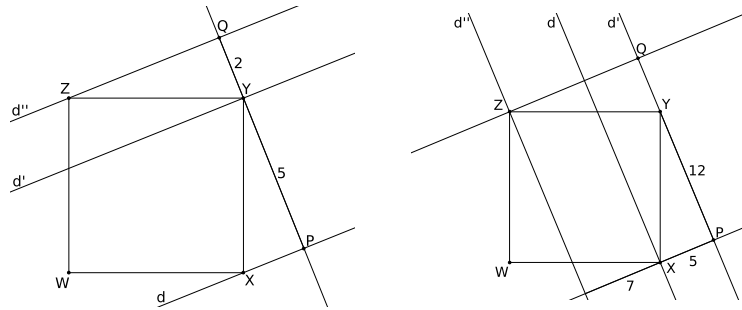
CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(4), p. 143–144.

CC166. Let $WXYZ$ be a square. Three parallel lines d, d' and d'' pass respectively through X, Y and Z . The distance between d and d' is 5 and the distance between d and d'' is 7. What is the area of the square?

Originally question 1 of the 2015 Midi Belgian mathematics contest.

The question as posed had two possible solutions. We received eight submissions, all of which correctly solved one of the two cases. The two cases are shown in the figure. We present the solution of John Heuver for the first case.



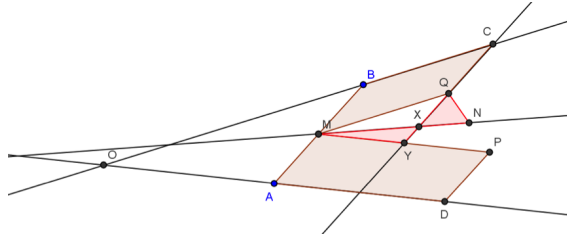
Draw a perpendicular line through Y meeting d' and d'' in P and Q respectively, then $|PY| = 5$ and $|QY| = 2$. Right angled triangles XPY and YQZ are congruent on account of alternating and complementary angles while $WXYZ$ is a square. Hence $|XP| = 2$ and thus from the triangle XPY we obtain $|XY| = \sqrt{29}$. Thus the area of $WXYZ$ is 29.

[Editor's comment. The area in the second case is 169.]

CC167. The lines BC and AD intersect at O , with both B and C on the same side of O , and the same goes for A and D . Among other properties, we have $|BC| = |AD|$, $2|OC| = 3|OB|$ and $|OD| = 2|OA|$. Points M and N are the respective middle points of segments $[AB]$ and $[CD]$. Quadrilaterals $ADPM$ and $BCQM$ are parallelograms. The line CQ cuts MN and MP respectively at X and Y . Show that triangles MXY and QXN have the same area.

Originally question 3 of the 2015 Midi Belgian mathematics contest.

We received four correct solutions. We present the solution by Titu Zvonaru.



Let the line MP intersect OC at R . Since M is the midpoint of AB and $MR \parallel AO$ by the definition of P , we get that $|RB| = |OB|/2$. Combining this with the information that $2|OC| = 3|OB|$ we get $|OR| = |RB| = |BC|$.

Let F be the intersection of the line CQ with OD . Since $CF \parallel BA$, we have that $|BC|/|OB| = |AF|/|OA|$. From the conditions given in the question $|BC|/|OB| = 1/2$ and $|OA| = |AD|$, so it follows that F is the midpoint of AD .

Consider $\triangle COF$. Since RY extends MY , we have $RY \parallel OF$, whence

$$\frac{|YC|}{|YF|} = \frac{|CR|}{|OR|} = 2.$$

Now consider the points A, Y and N which are on the extended sides of $\triangle CFD$ (by construction, N is the midpoint of CD). We have

$$\frac{|FA|}{|AD|} \cdot \frac{|DN|}{|NC|} \cdot \frac{|CY|}{|YF|} = \frac{1}{2} \cdot 1 \cdot 2 = 1.$$

By the converse Menelaus Theorem it follows that the points A, Y and N are collinear. Since A is the midpoint of OD and N is the midpoint of CD we have $AN \parallel OC$, which in turn, since $MQCB$ is a parallelogram and A, Y, N are collinear, means that $YN \parallel MQ$. This implies that $[MYQ] = [MNQ]$, which allows us to conclude that

$$[MXY] = [MYQ] - [MXQ] = [MNQ] - [MXQ] = [QXN],$$

as desired.

CC168. Six students from different European countries participate in an Erasmus course together. Each student speaks exactly two languages. Angela speaks German and English; Ulrich, German and Spanish; Carine, French and Spanish; Dieter, German and French; Pierre, French and English and Rocio Spanish and English. If we choose 2 people at random, what is the probability that they speak a common language?

Originally question 24 of the 2015 Primavera Mathematics Contest of Spain.

We received five submissions of which four were correct and complete. We present the solution by Henry Ricardo.

There are $6(5)/2 = 15$ ways to choose a pair of students at random. It is easy to see that only three of the 15 possible pairs *do not* share at least one language

in common – Angela-Carine, Ulrich-Pierre, and Dieter-Rocio. Thus the desired probability is

$$1 - 3/15 = 4/5.$$

CC169. What is the value of base b if

$$\log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} = 110.$$

Originally problem 17 of the 2015 Primavera Mathematics Contest of Spain.

We received eight complete and correct solutions, and present a composite of those solutions here.

Since $\log_b 10^k = k \log_b 10$ we have

$$\begin{aligned} \log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} &= 110 \\ (1 + 2 + \cdots + 10) \log_b 10 &= 110 \\ \frac{10 \cdot 11}{2} \log_b 10 &= 110 \\ \log_b 10 &= 2 \end{aligned}$$

so $b^2 = 10$, which gives $b = \sqrt{10}$ (discarding the negative option because a base cannot be negative).

CC170. The sum of 35 integers is S . We change 2 digits of one of the integers and the new sum is T . The difference $S - T$ is always divisible by which of the 5 numbers 2, 5, 7, 9 or 11?

Originally problem 15 of the 2015 Primavera Mathematics Contest of Spain.

We received six correct solutions. We present the solution of Hannes Geupel.

The difference $S - T$ is always divisible by 9. We only need to look at the number that gets changed in S and T , because the other numbers get subtracted by themselves in $S - T$. For example let this number be 21. $21 - 12 = 9$. 9 is not divisible by 2, 5, 7 or 11, so only the divisor 9 is left.

Now we prove that $S - T$ is always divisible by 9. Let N be a 2-digit number which is the one integer of S which gets changed. Let the first digit of N be a and the second digit be b . So $S - T = (10a + b) - (10b + a) = 9(a - b)$. Obviously, $9 \mid 9(a - b)$. In the general case, the two changed digits are $10^m a$ and $10^n b$ (the m^{th} and n^{th} digits are a and b respectively).

$S - T = (10^m a + 10^n b) - (10^m b + 10^n a) = (10^m - 10^n)(a - b)$. It is not hard to see that $10^m \equiv 1 \pmod{9}$ and $10^n \equiv 1 \pmod{9}$ so it follows that $10^m - 10^n \equiv 0 \pmod{9}$. Hence $(10^m - 10^n)(a - b) \equiv 0 \pmod{9}$ or rather, $9 \mid (10^m - 10^n)(a - b) = S - T$.

