

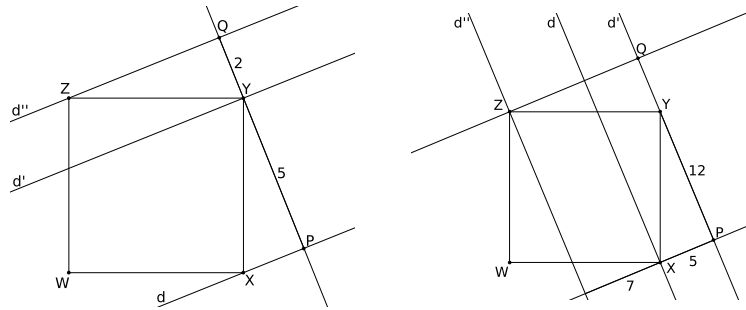
# CONTEST CORNER SOLUTIONS

*Les énoncés des problèmes dans cette section paraissent initialement dans 2015: 41(4), p. 143–144.*

**CC166.** Let  $WXYZ$  be a square. Three parallel lines  $d, d'$  and  $d''$  pass respectively through  $X, Y$  and  $Z$ . The distance between  $d$  and  $d'$  is 5 and the distance between  $d$  and  $d''$  is 7. What is the area of the square?

*Originally question 1 of the 2015 Midi Belgian mathematics contest.*

*The question as posed had two possible solutions. We received eight submissions, all of which correctly solved one of the two cases. The two cases are shown in the figure. We present the solution of John Heuver for the first case.*



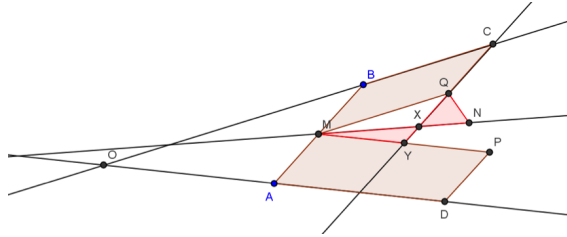
Draw a perpendicular line through  $Y$  meeting  $d'$  and  $d''$  in  $P$  and  $Q$  respectively, then  $|PY| = 5$  and  $|QY| = 2$ . Right angled triangles  $XPY$  and  $YQZ$  are congruent on account of alternating and complementary angles while  $WXYZ$  is a square. Hence  $|XP| = 2$  and thus from the triangle  $XPY$  we obtain  $|XY| = \sqrt{29}$ . Thus the area of  $WXYZ$  is 29.

*[Editor's comment. The area in the second case is 169.]*

**CC167.** The lines  $BC$  and  $AD$  intersect at  $O$ , with both  $B$  and  $C$  on the same side of  $O$ , and the same goes for  $A$  and  $D$ . Among other properties, we have  $|BC| = |AD|$ ,  $2|OC| = 3|OB|$  and  $|OD| = 2|OA|$ . Points  $M$  and  $N$  are the respective middle points of segments  $[AB]$  and  $[CD]$ . Quadrilaterals  $ADPM$  and  $BCQM$  are parallelograms. The line  $CQ$  cuts  $MN$  and  $MP$  respectively at  $X$  and  $Y$ . Show that triangles  $MXY$  and  $QXN$  have the same area.

*Originally question 3 of the 2015 Midi Belgian mathematics contest.*

*We received four correct solutions. We present the solution by Titu Zvonaru.*



Let the line  $MP$  intersect  $OC$  at  $R$ . Since  $M$  is the midpoint of  $AB$  and  $MR \parallel AO$  by the definition of  $P$ , we get that  $|RB| = |OB|/2$ . Combining this with the information that  $2|OC| = 3|OB|$  we get  $|OR| = |RB| = |BC|$ .

Let  $F$  be the intersection of the line  $CQ$  with  $OD$ . Since  $CF \parallel BA$ , we have that  $|BC|/|OB| = |AF|/|OA|$ . From the conditions given in the question  $|BC|/|OB| = 1/2$  and  $|OA| = |AD|$ , so it follows that  $F$  is the midpoint of  $AD$ .

Consider  $\triangle COF$ . Since  $RY$  extends  $MY$ , we have  $RY \parallel OF$ , whence

$$\frac{|YC|}{|YF|} = \frac{|CR|}{|OR|} = 2.$$

Now consider the points  $A, Y$  and  $N$  which are on the extended sides of  $\triangle CFD$  (by construction,  $N$  is the midpoint of  $CD$ ). We have

$$\frac{|FA|}{|AD|} \cdot \frac{|DN|}{|NC|} \cdot \frac{|CY|}{|YF|} = \frac{1}{2} \cdot 1 \cdot 2 = 1.$$

By the converse Menelaus Theorem it follows that the points  $A, Y$  and  $N$  are collinear. Since  $A$  is the midpoint of  $OD$  and  $N$  is the midpoint of  $CD$  we have  $AN \parallel OC$ , which in turn, since  $MQCB$  is a parallelogram and  $A, Y, N$  are collinear, means that  $YN \parallel MQ$ . This implies that  $[MYQ] = [MNQ]$ , which allows us to conclude that

$$[MXY] = [MYQ] - [MXQ] = [MNQ] - [MXQ] = [QXN],$$

as desired.

**CC168.** Six students from different European countries participate in an Erasmus course together. Each student speaks exactly two languages. Angela speaks German and English; Ulrich, German and Spanish; Carine, French and Spanish; Dieter, German and French; Pierre, French and English and Rocio Spanish and English. If we choose 2 people at random, what is the probability that they speak a common language?

*Originally question 24 of the 2015 Primavera Mathematics Contest of Spain.*

*We received five submissions of which four were correct and complete. We present the solution by Henry Ricardo.*

There are  $6(5)/2 = 15$  ways to choose a pair of students at random. It is easy to see that only three of the 15 possible pairs *do not* share at least one language

in common – Angela-Carine, Ulrich-Pierre, and Dieter-Rocio. Thus the desired probability is

$$1 - 3/15 = 4/5.$$

**CC169.** What is the value of base  $b$  if

$$\log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} = 110.$$

*Originally problem 17 of the 2015 Primavera Mathematics Contest of Spain.*

*We received eight complete and correct solutions, and present a composite of those solutions here.*

Since  $\log_b 10^k = k \log_b 10$  we have

$$\begin{aligned} \log_b 10 + \log_b 10^2 + \cdots + \log_b 10^{10} &= 110 \\ (1 + 2 + \cdots + 10) \log_b 10 &= 110 \\ \frac{10 \cdot 11}{2} \log_b 10 &= 110 \\ \log_b 10 &= 2 \end{aligned}$$

so  $b^2 = 10$ , which gives  $b = \sqrt{10}$  (discarding the negative option because a base cannot be negative).

**CC170.** The sum of 35 integers is  $S$ . We change 2 digits of one of the integers and the new sum is  $T$ . The difference  $S - T$  is always divisible by which of the 5 numbers 2, 5, 7, 9 or 11?

*Originally problem 15 of the 2015 Primavera Mathematics Contest of Spain.*

*We received six correct solutions. We present the solution of Hannes Geupel.*

The difference  $S - T$  is always divisible by 9. We only need to look at the number that gets changed in  $S$  and  $T$ , because the other numbers get subtracted by themselves in  $S - T$ . For example let this number be 21.  $21 - 12 = 9$ . 9 is not divisible by 2, 5, 7 or 11, so only the divisor 9 is left.

Now we prove that  $S - T$  is always divisible by 9. Let  $N$  be a 2-digit number which is the one integer of  $S$  which gets changed. Let the first digit of  $N$  be  $a$  and the second digit be  $b$ . So  $S - T = (10a + b) - (10b + a) = 9(a - b)$ . Obviously,  $9 \mid 9(a - b)$ . In the general case, the two changed digits are  $10^m a$  and  $10^n b$  (the  $m^{\text{th}}$  and  $n^{\text{th}}$  digits are  $a$  and  $b$  respectively).

$S - T = (10^m a + 10^n b) - (10^m b + 10^n a) = (10^m - 10^n)(a - b)$ . It is not hard to see that  $10^m \equiv 1 \pmod{9}$  and  $10^n \equiv 1 \pmod{9}$  so it follows that  $10^m - 10^n \equiv 0 \pmod{9}$ . Hence  $(10^m - 10^n)(a - b) \equiv 0 \pmod{9}$  or rather,  $9 \mid (10^m - 10^n)(a - b) = S - T$ .

