

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2015: 41(3), p. 119–123.*

**4021.** *Proposed by Arkady Alt.*

Let  $(\bar{a}_n)_{n \geq 0}$  be a sequence of Fibonacci vectors defined recursively by  $\bar{a}_0 = \bar{a}$ ,  $\bar{a}_1 = \bar{b}$  and  $\bar{a}_{n+1} = \bar{a}_n + \bar{a}_{n-1}$  for all integers  $n \geq 1$ . Prove that, for all integers  $n \geq 1$ , the sum of vectors  $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1}$  equals  $k\bar{a}_i$  for some  $i$  and constant  $k$ .

*We received nine correct solutions. We present the solution by David Stone and John Hawkins (joint).*

We shall prove that  $\bar{a}_0 + \bar{a}_1 + \cdots + \bar{a}_{4n+1} = L_{2n+1}\bar{a}_{2n+2}$ , where  $L_k$  is the  $k$ th Lucas number. We use some easily proven results. Here,  $F_k$  is the  $k$ th Fibonacci number.

1.  $F_0 + F_1 + \cdots + F_m = F_{m+2} - 1$ .
2.  $F_{4n+2} = L_{2n+1}F_{2n+1}$
3.  $F_{4n+3} = L_{2n+1}F_{2n+2} + 1$
4.  $\bar{a}_k = F_{k-1}\bar{a}_0 + F_k\bar{a}_1$  for  $k \geq 1$ .

Therefore,

$$\begin{aligned} \sum_{k=0}^m \bar{a}_k &= \bar{a}_0 + \sum_{k=1}^m (F_{k-1}\bar{a}_0 + F_k\bar{a}_1) \\ &= \bar{a}_0 + \left( \sum_{k=1}^m F_{k-1} \right) \bar{a}_0 + \left( \sum_{k=1}^m F_k \right) \bar{a}_1 \\ &= \bar{a}_0 + (F_{m+1} - 1)\bar{a}_0 + (F_{m+2} - 1)\bar{a}_1 \\ &= F_{m+1}\bar{a}_0 + F_{m+2}\bar{a}_1 - \bar{a}_1 \\ &= \bar{a}_{m+2} - \bar{a}_1 \end{aligned}$$

Hence, with  $m = 4n + 1$ ,

$$\begin{aligned} \sum_{k=0}^{4n+1} \bar{a}_k &= \bar{a}_{4n+3} - \bar{a}_1 = F_{4n+2}\bar{a}_0 + F_{4n+3}\bar{a}_1 - \bar{a}_1 \\ &= (L_{2n+1}F_{2n+1})\bar{a}_0 + (L_{2n+1}F_{2n+2})\bar{a}_1 \\ &= L_{2n+1}(F_{2n+1}\bar{a}_0 + F_{2n+2}\bar{a}_1) \\ &= L_{2n+1}\bar{a}_{2n+2}. \end{aligned}$$

*Editor's Comments.* Various solvers expressed the coefficient of  $\bar{a}_{2n+2}$  as  $L_{2n+1}$ ,  $F_{2n} + F_{2n+2}$ , and  $\frac{F_{4n+2}}{F_{2n+1}}$  and variations of these resulting from different indexing of the Fibonacci sequence. Swylan pointed out that if the word 'constant' is interpreted to mean 'independent of  $n$ ', then the claim of the problem is false. Perhaps 'scalar' would have been a better word.

**4022.** *Proposed by Leonard Giugiuc.*

In a triangle  $ABC$ , let internal angle bisectors from angles  $A, B$  and  $C$  intersect the sides  $BC, CA$  and  $AB$  in points  $D, E$  and  $F$  and let the incircle of  $\triangle ABC$  touch the sides in  $M, N$ , and  $P$ , respectively. Show that

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

*We received eleven submissions, of which seven were correct, two were incorrect, and two were incomplete. We present the solution by Titu Zvonaru.*

Define  $x = NA = PA$ ,  $y = PB = MB$ , and  $z = MC = NC$ ; then

$$BC = y + z, \quad CA = z + x, \quad \text{and} \quad AB = x + y.$$

By the angle bisector theorem we have

$$\frac{FA}{FB} = \frac{CA}{BC} = \frac{z+x}{y+z}, \quad \frac{DB}{DC} = \frac{AB}{CA} = \frac{x+y}{z+x}, \quad \text{and} \quad \frac{EC}{EA} = \frac{BC}{AB} = \frac{y+z}{x+y}.$$

We therefore have to prove that for positive real numbers  $x, y, z$ ,

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{z+x}{y+z} + \frac{x+y}{z+x} + \frac{y+z}{x+y}. \quad (1)$$

After clearing denominators what we must prove reduces to

$$x^2y^4 + y^2z^4 + z^2x^4 + x^3y^3 + y^3z^3 + z^3x^3 \geq x^3yz^2 + x^2y^3z + xy^2z^3 + 3x^2y^2z^2. \quad (2)$$

By the AM-GM inequality we have

$$\begin{aligned} x^2y^4 + y^2z^4 + z^2x^4 &\geq 3x^2y^2z^2, \\ x^3y^3 + z^3x^3 + z^3x^3 &\geq 3x^3yz^2, \\ y^3z^3 + x^3y^3 + x^3y^3 &\geq 3x^2y^3z, \quad \text{and} \\ z^3x^3 + y^3z^3 + y^3z^3 &\geq 3xy^2z^3, \end{aligned}$$

which together imply that (2) holds. Equality holds if and only if  $x = y = z$ , which immediately implies that the triangle is equilateral.

*Editor's Comments.* Most submissions reduced our problem to equation (1), but then algebra caused difficulties with two of the faulty arguments. The solution from Salem Malikić neatly avoided calculations by remarking that (1) is known;

see, for example, the Belarussian IMO Team preparation tests of 1997, where calculations are much simplified by exploiting the cyclic symmetry of the inequalities. Beware, however, that one must not assume noncyclic symmetry (as in one of the incomplete submissions).

**4023.** *Proposed by Ali Behrouz.*

Find all functions  $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$  such that for all  $x, y \in \mathbb{R}$  with  $x > y$ , we have

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)).$$

*We received two correct submissions. We present the solution by Joseph Ling.*

It is easy to see that  $f(x) = \frac{1}{x} \forall x > 0$  satisfies

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)) \quad (1)$$

whenever  $0 < y < x$ . We show that there are no other solutions  $f : (0, \infty) \rightarrow (0, \infty)$  to (1).

First, we note that  $f$  is one-to-one. For if  $0 < y < x$  are such that  $f(y) = f(x)$ , then (1) implies that  $f\left(\frac{x}{x-y}\right) = 0$ , which is impossible.

Second, we note that  $f(x) \leq \frac{1}{x}$  for all  $x > 0$ . For if there exists  $x > 0$  such that  $f(x) > \frac{1}{x}$ , then  $y = x - \frac{1}{f(x)}$  satisfies  $0 < y < x$ . But then  $\frac{x}{x-y} = xf(x)$  and (1) will imply that  $f(xf(y)) = 0$ , which is impossible.

Now, suppose that  $0 < y_1 < y_2$ . Consider  $x = y_2 + \frac{1}{f(y_1)}$ . Then  $0 < y_1 < y_2 < x$  and (1) implies that

$$f\left(\frac{x}{x-y_2}\right) + f(xf(y_2)) = f(xf(x)) = f\left(\frac{x}{x-y_1}\right) + f(xf(y_1)). \quad (2)$$

By the definition of  $x$ ,  $\frac{x}{x-y_2} = xf(y_1)$ . So, (2) is reduced to  $f(xf(y_2)) = f\left(\frac{x}{x-y_1}\right)$ . Since  $f$  is one-to-one, we have  $xf(y_2) = \frac{x}{x-y_1}$ , and so,  $x = y_1 + \frac{1}{f(y_2)}$ . Using this and the definition of  $x$ , we see that  $\frac{1}{f(y_1)} - y_1 = \frac{1}{f(y_2)} - y_2$ . Since  $y_1$  and  $y_2$  are arbitrary,  $\frac{1}{f(y)} - y$  is independent of  $y$ , and so, it must be some constant, say,  $c$ . That is,

$$f(y) = \frac{1}{y+c}$$

for all  $y > 0$ .

It remains to show that  $c = 0$ . Since  $f(x) \leq \frac{1}{x}$  for all  $x$ ,  $c \geq 0$ . Furthermore, if

$c > 0$ , then for any  $0 < y < x$ , we have

$$\begin{aligned} f\left(\frac{x}{x-y}\right) + f(xf(y)) &= \frac{x-y}{x+c(x-y)} + \frac{y+c}{x+c(y+c)} \\ &> \frac{x-y}{x+c(x+c)} + \frac{y+c}{x+c(x+c)} = \frac{x+c}{x+c(x+c)} \\ &= f(xf(x)), \end{aligned}$$

a contradiction to (1). So,  $c = 0$  and our proof is complete.

**4024.** *Proposed by Leonard Giugiuc.*

Let  $a, b, c$  and  $d$  be real numbers such that  $a^2 + b^2 + c^2 + d^2 = 4$ . Prove that

$$abc + abd + acd + bcd + 4 \geq a + b + c + d$$

and determine when equality holds.

*We received three correct solutions. We present the solution by Titu Zvononu, modified by the editor.*

We consider several cases separately.

*Case 1.* If  $a, b, c, d \geq 0$ , then by the Cauchy-Schwarz Inequality, we have

$$(a + b + c + d)^2 \leq (1^2 + 1^2 + 1^2 + 1^2)(a^2 + b^2 + c^2 + d^2) = 16.$$

Thus,  $a + b + c + d \leq 4$  from which the result follows immediately.

*Case 2.* If  $a, b, c, d \leq 0$ , we set  $x = -a$ ,  $y = -b$ ,  $z = -c$ , and  $t = -d$ . Then,  $x, y, z, t \geq 0$  and we would now like to show that

$$\begin{aligned} -(xyz + xyt + xzt + yzt) + 4 &\geq -(x + y + z + t) \quad \text{or} \\ 2(x + y + z + t) - (xyz + xyt + xzt + yzt) + 4 &\geq x + y + z + t. \end{aligned} \quad (1)$$

Since we know  $x^2 + y^2 + z^2 + t^2 = 4$ , we have by the first case (with  $a, b, c, d$  replaced with  $x, y, z, t$ , respectively and symbolically), that

$$x + y + z + t \leq 4. \quad (2)$$

Then,

$$\begin{aligned} &4(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= (x^2 + y^2 + z^2 + t^2)(x + y + z + t) - 2(xyz + xyt + xzt + yzt) \\ &= x(y - z)^2 + y(x - t)^2 + z(x - t)^2 + t(y - z)^2 + x(x^2 + t^2) \\ &\quad + y(y^2 + z^2) + z(z^2 + y^2) + t(t^2 + x^2) \\ &\geq 0, \end{aligned}$$

which together with (2) implies (1).

*Case 3.* If one of  $a, b, c, d$  is nonnegative and the other three are nonpositive. Due to the symmetry in the given equation and the one we wish to prove, we may assume that  $a \geq 0$  and  $b, c, d \leq 0$ . Here we let  $y = -b$ ,  $z = -c$ , and  $t = -d$ . Then,  $a, y, z, t \geq 0$  with  $a^2 + y^2 + z^2 + t^2 = 4$  and we wish to prove  $ayz + ayt + azt - yzt + 4 \geq a - y - z - t$  or

$$2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq a + y + z + t. \quad (3)$$

Now,

$$\begin{aligned} & 4(y + z + t) + 2(ayz + ayt + azt - yzt) \\ &= (a^2 + y^2 + z^2 + t^2)(y + z + t) + 2(ayz + ayt + azt - yzt) \\ &\geq y(z^2 + t^2) - 2yzt \\ &= y(z - t)^2 \\ &\geq 0, \end{aligned}$$

so,  $2(y + z + t) + ayz + ayt + azt - yzt + 4 \geq 4$ , thus, establishing (3), as desired, since  $a + y + z + t \leq 4$  (see Case 1).

*Case 4.* If  $a, b \geq 0$  and  $c, d \leq 0$ , we set  $z = -c$  and  $t = -d$ . Then  $a, b, z, t \geq 0$  such that  $a^2 + b^2 + z^2 + t^2 = 4$  and we would like to show that

$$\begin{aligned} & -abz - abt + azt + bzt + 4 \geq a + b - z - t \quad \text{or} \\ & 2(z + t) - abz - abt + azt + bzt + 4 \geq a + b + z + t. \end{aligned} \quad (4)$$

Now,

$$\begin{aligned} & 4(z + t) - 2(abz + abt - azt - bzt) \\ &= (a^2 + b^2 + z^2 + t^2)(z + t) - 2(abz + abt - azt - bzt) \\ &= z(a - b)^2 + t(a - b)^2 + (z + t)(z^2 + t^2) + 2azt + 2bzt \\ &\geq 0. \end{aligned}$$

So,  $2(z + t) - abz - abt + azt + bzt \geq 4$ , thus establishing (4) since  $a + b + z + t \leq 4$ .

*Case 5.* If  $a, b, c \geq 0$  and  $d \leq 0$ , we set  $t = -d$ . Then  $a, b, c, t \geq 0$  with  $a^2 + b^2 + c^2 + t^2 = 4$  and we would like to show that

$$\begin{aligned} & abc - abt - act - bct + 4 \geq a + b + c - t \quad \text{or} \\ & 2t + abc - abt - act - bct + 4 \geq a + b + c + t. \end{aligned} \quad (5)$$

Since  $a + b + c + t \leq 4$ , to establish (5), it suffices to show that

$$(a^2 + b^2 + c^2 + t^2)t + 2(abc - abt - act - bct) \geq 0. \quad (6)$$

We let  $L$  denote the left-hand side of (6) and assume, without loss of generality, that  $a \geq b \geq c$ . Note that

$$L = t(b-c)^2 + t(a-t)^2 + 2a(t-b)(t-c) \quad (7)$$

$$\text{and } L = t(a-b)^2 + t(c-t)^2 + 2c(t-a)(t-b). \quad (8)$$

If  $t \leq b$ , then from (7) we can see that  $L \geq 0$  and if  $t \geq b$ , then  $L \geq 0$  from (8). Hence, we can conclude that (6) is true, as desired.

Examining the five cases, it is readily seen that equality can only hold in Case 5 when  $a = b = c = t$ ; that is, if and only if  $(a, b, c, d) = (1, 1, 1, -1)$  and all its permutations.

**4025.** *Proposed by Dragoljub Milošević.*

Prove that for positive numbers  $a, b$  and  $c$ , we have

$$\sqrt[3]{\left(\frac{a}{2b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{2c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{2a+b}\right)^2} \geq \sqrt[3]{3}.$$

*We received eleven correct solutions. We present the solution by Salem Madikić.*

Let  $f(a, b, c)$  denote the left hand side of the given inequality. By the AM-GM Inequality, we have

$$\begin{aligned} \sqrt[3]{\left(\frac{a}{2b+c}\right)^2} &= \frac{a}{\sqrt[3]{a(2b+c)^2}} = \frac{a}{\sqrt[3]{9} \sqrt[3]{a \cdot \frac{2b+c}{3} \cdot \frac{2b+c}{3}}} \\ &\geq \frac{3a}{\sqrt[3]{9} \left(a + \frac{2b+c}{3} + \frac{2b+c}{3}\right)} \\ &= \frac{3\sqrt[3]{3}a}{3a + 4b + 2c}. \end{aligned}$$

Using similar inequalities involving the other two summands, we then have

$$f(a, b, c) \geq 3\sqrt[3]{3} \sum_{\text{cyc}} \frac{a}{3a + 4b + 2c}. \quad (1)$$

Now, by the Cauchy-Schwarz Inequality, we have

$$\left(\sum \left(\sqrt{\frac{a}{3a+4b+2c}}\right)^2\right) \left(\sum \left(\sqrt{a(3a+4b+2c)}\right)^2\right) \geq \sum a^2.$$

So,

$$\sum \frac{a}{3a+4b+2c} \geq \frac{\sum a^2}{\sum a(3a+4b+2c)} = \frac{\sum a^2}{3\sum a^2 + 6\sum ab} = \frac{1}{3}. \quad (2)$$

Substituting (2) into (1),  $f(a, b, c) \geq \sqrt[3]{3}$  follows immediately.

To achieve equality, we must have  $3a = 2b + c$ ,  $3b = 2c + a$ , and  $3c = 2a + b$ . Without loss of generality, we may assume that  $\max\{a, b, c\} = a$ . Then,  $3a = 2b + c$  implies  $2(a - b) + (a - c) = 0$ , so  $a = b = c$ . Conversely, it is readily checked that if  $a = b = c$ , then equality holds.

*Editor's Comments.* Using convexity and Jensen's Inequality, Stadler proved that in general,  $\sum \left(\frac{a}{2b+c}\right)^k \geq 3^{1-k}$  for all  $k \geq 0$ .

**4026.** *Proposed by Roy Barbara.*

Prove or disprove the following property: if  $r$  is any non-zero rational number, then the real number  $x = (1+r)^{1/3} + (1-r)^{1/3}$  is irrational.

*We received two correct solutions. We present the solution by Joseph DiMuro.*

Assume both  $r$  and  $x$  are rational numbers with  $r \neq 0$ . Setting  $y_1 = (1+r)^{1/3}$  and  $y_2 = (1-r)^{1/3}$  we can show that

$$\frac{x^3 - 2}{3x} = y_1 y_2.$$

That means that  $y_1$  and  $y_2$  are the two solutions for  $y$  of  $y^2 - xy + \frac{x^3-2}{3x} = 0$ . But using the quadratic formula, we also obtain

$$y = \frac{x \pm \sqrt{x^2 - \frac{4x^3-8}{3x}}}{2} = \frac{x}{2} \pm \sqrt{\frac{8-x^3}{12x}}.$$

This shows that if  $x$  is rational then  $y_1$  and  $y_2$  are contained in quadratic extensions of  $\mathbb{Q}$ . On the other hand, if  $r$  is rational then  $y_1 = (1+r)^{1/3}$  and  $y_2 = (1-r)^{1/3}$  are contained in cubic extensions of  $\mathbb{Q}$  as well. Both of these can only be true if  $y_1$  and  $y_2$  are rational numbers themselves.

Let  $r = \frac{a}{b}$ , where  $a, b$  are relatively prime non-zero integers. Then  $y_1 = (\frac{b+a}{b})^{1/3}$  and  $y_2 = (\frac{b-a}{b})^{1/3}$ . The fractions  $\frac{b+a}{b}$  and  $\frac{b-a}{b}$  are in lowest terms, so for them to be perfect cubes, their numerators and denominators must be perfect cubes. Then we have an arithmetic progression  $b-a, b, b+a$  of cubes, which is known to be impossible (e.g. see P. Dénes, Über die Diophantische Gleichung  $x^l + y^l = cz^l$ , *Acta. Math.* **88** (1952) 241-251).

*Editor's Comments.* The statement that there is no arithmetic progression of three cubes can be proven with elementary number theory and is an interesting exercise.

**4027.** *Proposed by George Apostolopoulos.*

Let  $a, b$  and  $c$  be positive real numbers such that  $a + b + c = 3$ . Prove that

$$\frac{ab}{a+ab+b} + \frac{bc}{b+bc+c} + \frac{ac}{a+ac+c} \leq 1.$$

*We received 24 submissions of which 22 were correct and complete. We present 5 solutions, each of them insightful in a different way.*

*Solution 1, by Ali Adnan.*

Observe that

$$\sum_{cyc} \frac{ab}{a+ab+b} \leq 1 \iff \sum_{cyc} \frac{9}{\frac{a+b}{ab}+1} \leq 9. \quad (1)$$

Now, from Cauchy-Schwarz Inequality,

$$\frac{9}{\frac{a+b}{ab}+1} = \frac{9}{\frac{1}{a}+\frac{1}{b}+1} \leq a+b+1,$$

and adding up analogous such inequalities cyclically, (1) follows.

*Solution 2, by Ali Adnan.*

We note that the inequality is equivalent to

$$\sum_{cyc} \frac{a+b}{a+b+ab} \geq 2 \iff \sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \geq 1,$$

which follows easily from the AM-HM and Cauchy-Schwarz Inequalities:

$$\sum_{cyc} \frac{1}{2+\frac{2ab}{a+b}} \geq \sum_{cyc} \frac{1}{2+\frac{a+b}{2}} \geq \frac{(1+1+1)^2}{6+a+b+c} = 1,$$

thus completing the proof.

*Solution 3, by Henry Ricardo.*

We have

$$\begin{aligned} \sum_{cyclic} \frac{ab}{a+ab+b} &= \sum_{cyclic} \frac{1}{\frac{1}{b}+1+\frac{1}{a}} = \frac{1}{3} \sum_{cyclic} \frac{3}{\frac{1}{b}+1+\frac{1}{a}} \\ &\leq \frac{1}{3} \sum_{cyclic} \frac{a+b+1}{3} = \frac{1}{3} \left( \frac{2(a+b+c)+3}{3} \right) = 1, \end{aligned}$$

where we have used the harmonic mean-arithmetic mean inequality.

Equality holds if and only if  $a = b = c = 1$ .

*Solution 4, by Salem Malikić.*

Using the inequality between arithmetic and geometric mean for positive reals  $x$  and  $y$  we have

$$x+xy+y \geq 3\sqrt[3]{x^2y^2}$$

with equality if and only if  $x = xy = y$ , that gives  $x = y = 1$  and implying that

$$\frac{xy}{x+xy+y} \leq \frac{\sqrt[3]{xy}}{3}.$$



Using this inequality we have

$$\frac{ab}{a+ab+b} + \frac{bc}{b+bc+c} + \frac{ca}{c+ca+a} \leq \frac{\sqrt[3]{ab} + \sqrt[3]{bc} + \sqrt[3]{ca}}{3}.$$

Now, using Power-mean inequality, we have

$$\frac{\sqrt[3]{ab} + \sqrt[3]{bc} + \sqrt[3]{ca}}{3} \leq \sqrt[3]{\frac{ab+bc+ca}{3}} \leq \sqrt[3]{\frac{(a+b+c)^2}{3}} = 1.$$

where in the last step we used the well known inequality

$$3(ab+bc+ca) \leq (a+b+c)^2.$$

This completes our proof.

In order to achieve equality we must have  $a = b = c = 1$ . It is easy to verify that this is indeed an equality case.

*Solution 5, by Leonard Giugiuc.*

The inequality is equivalent to

$$\frac{1}{\frac{1}{a} + \frac{1}{b} + 1} + \frac{1}{\frac{1}{b} + \frac{1}{c} + 1} + \frac{1}{\frac{1}{c} + \frac{1}{a} + 1} \leq 1.$$

By AM-HM Inequality,

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}, \quad \frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c}, \quad \frac{1}{c} + \frac{1}{a} \geq \frac{4}{c+a}.$$

From here,

$$\frac{1}{\frac{1}{a} + \frac{1}{b} + 1} + \frac{1}{\frac{1}{b} + \frac{1}{c} + 1} + \frac{1}{\frac{1}{c} + \frac{1}{a} + 1} \leq \frac{1}{\frac{4}{a+b} + 1} + \frac{1}{\frac{4}{b+c} + 1} + \frac{1}{\frac{4}{c+a} + 1}.$$

But

$$\frac{1}{\frac{4}{a+b} + 1} + \frac{1}{\frac{4}{b+c} + 1} + \frac{1}{\frac{4}{c+a} + 1} = \frac{a+b}{a+b+4} + \frac{b+c}{b+c+4} + \frac{c+a}{c+a+4}.$$

Since the function  $f(x) = \frac{x}{x+4}$  is concave if  $x > 0$ , then by Jensen's Inequality we get

$$f(a+b) + f(b+c) + f(c+a) \leq 3f\left(\frac{2(a+b+c)}{3}\right) = 3f(2) = 1.$$

So,

$$\frac{a+b}{a+b+4} + \frac{b+c}{b+c+4} + \frac{c+a}{c+a+4} \leq 1.$$

**4028.** *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line  $\ell$  meets orthogonally two distinct parallel planes  $\mathcal{P}$  and  $\mathcal{P}'$  at  $H$  and  $H'$ . Let  $r$  and  $r'$  be positive real numbers with  $r \leq r'$ ; let  $\mathcal{C}$  be the circle in  $\mathcal{P}$  with center  $H$ , radius  $r$ , and let  $\mathcal{C}'$  in  $\mathcal{P}'$  be similarly defined. For a fixed point  $M'$  on  $\mathcal{C}'$ , find the maximum distance between the lines  $\ell$  and  $MM'$  as  $M$  moves about the circle  $\mathcal{C}$  (where the distance between two lines is the minimum distance from a point of one line to a point of the other).

*We received four correct solutions and will feature two of them that are quite similar except that the first makes use of coordinates.*

*Solution 1, by Oliver Geupel.*

We prove that the required maximum distance is  $r$ . We use Cartesian coordinates such that  $H' = (0, 0, 0)$ ,  $M' = (r', 0, 0)$ , and  $H = (0, 0, h)$  where  $h \in \mathbb{R}$ . For every point  $M$  on  $\mathcal{C}$ , the distance between  $\ell$  and  $MM'$  is not greater than  $|MH| = r$  (because that distance is, by definition, the length of the shortest among all segments joining a point of  $\ell$  to a point of  $MM'$ , which is therefore at most  $|MH|$ ). Moreover, the distance between two non-intersecting lines is measured along a line that is perpendicular to both. Put

$$M = \left( \frac{r}{r'} \cdot r, \sqrt{1 - \frac{r^2}{r'^2}} \cdot r, h \right),$$

which is on  $\mathcal{C}$ . We have  $\overrightarrow{HM} \cdot \overrightarrow{HH'} = 0$  and

$$\overrightarrow{M'M} \cdot \overrightarrow{HM} = \left( \frac{r^2}{r'} - r', \sqrt{1 - \frac{r^2}{r'^2}} \cdot r, h \right) \cdot \left( \frac{r^2}{r'}, \sqrt{1 - \frac{r^2}{r'^2}} \cdot r, 0 \right) = 0,$$

so that  $HM$  is perpendicular to both  $\ell$  and  $MM'$ . Therefore the distance between the lines  $\ell$  and  $MM'$  is  $|HM| = r$ , which completes the proof.

*Solution 2, by Edmund Swylan.*

For every point  $M$  on  $\mathcal{C}$ , let  $\mathcal{Q}$  be the plane orthogonal to  $\ell$  that contains a point  $P$  of  $MM'$  nearest to  $\ell$ . Let  $O, \mathcal{D}, \mathcal{D}', N, N'$  be the orthogonal projections of  $\ell, \mathcal{C}, \mathcal{C}', M, M'$ , respectively, onto  $\mathcal{Q}$ . The distance between the lines  $\ell$  and  $MM'$  projects to  $|PO|$ . Our problem is thereby reduced to a 2-dimensional problem:

Given circles  $\mathcal{D}$  and  $\mathcal{D}'$  in the same plane with common centre  $O$  and radii  $r$  and  $r'$ , a fixed point  $N'$  on  $\mathcal{D}'$ , and a point  $N$  moving about  $\mathcal{D}$ , what is the maximum distance from  $O$  to  $NN'$ ?

The answer is  $r$ .

For  $r < r'$  the maximum is achieved if and only if  $NN'$  is tangent to  $\mathcal{D}$ . For  $r = r'$  it is achieved if and only if  $N = N'$  and the line  $NN'$  degenerates into a point which occurs when  $MM'$  is parallel to  $\ell$ .

**4029.** *Proposed by Paul Bracken.*

Suppose  $a > 0$ . Find the solutions of the following equation in the interval  $(0, \infty)$ :

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a.$$

*We received four correct solutions and will feature two different ones.*

*Solution 1. We present a composite of the very similar solutions by Arkady Alt and the proposer, Paul Bracken. Another similar solution was received from Oliver Geupel.*

It is clear that

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)}, \quad \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)(x+2)} = \frac{2}{x(x+1)(x+2)},$$

and

$$\begin{aligned} \frac{n!}{x(x+1)(x+2)\cdots(x+n)} - \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} \\ = \frac{(n+1)!}{x(x+1)(x+2)\cdots(x+n+1)}. \end{aligned}$$

It therefore follows by induction that

$$\frac{1}{x} - \frac{1}{x+1} - \sum_{k=1}^{n-1} \frac{k!}{(x+1)(x+2)\cdots(x+k+1)} = \frac{n!}{x(x+1)(x+2)\cdots(x+n)}.$$

However, for  $x > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{n!}{x(x+1)\cdots(x+n)} = 0,$$

since

$$\frac{n!}{x(x+1)(x+2)\cdots(x+n)} = \frac{1}{x(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right)}$$

and

$$(x+1)\left(\frac{x}{2}+1\right)\cdots\left(\frac{x}{n}+1\right) > 1+x\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right).$$

Hence the left-hand side of the original equation is given by

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = \frac{1}{x}.$$

Therefore the original equation is equivalent to  $x^2 - ax - 1 = 0$ . This quadratic equation has the following unique solution in  $(0, \infty)$ :

$$x_r = \frac{1}{2}(a + \sqrt{a^2 + 4}).$$

*Solution 2, by Albert Stadler.*

We note that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} \\
 &= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+1)(x+2)\cdots(x+n+1)} \\
 &= \sum_{n=1}^{\infty} \frac{\Gamma(x+1)\Gamma(n+1)}{\Gamma(x+n+1)} \\
 &= \sum_{n=1}^{\infty} \beta(x+1, n+1) \\
 &= \sum_{n=1}^{\infty} \int_0^1 t^x (1-t)^n dt \\
 &= \int_0^1 t^{x-1} (1-t) dt \\
 &= \frac{1}{x} - \frac{1}{x+1}, \quad x > 0.
 \end{aligned}$$

The original equation is therefore equivalent to  $\frac{1}{x} = x - a$  or  $x^2 - ax - 1 = 0$ . This quadratic equation has exactly one positive root, which is  $x_r = \frac{1}{2}(a + \sqrt{a^2 + 4})$ .

**4030.** *Proposed by Paolo Perfetti.*

a) Prove that  $4^{\cos t} + 4^{\sin t} \geq 5$  for  $t \in [0, \frac{\pi}{4}]$ .

b) Prove that  $6^{\cos t} + 6^{\sin t} \geq 7$  for  $t \in [0, \frac{\pi}{4}]$ .

*There were six submitted solutions for this problem, four of which were correct. We present the solution by Michel Bataille.*

*Lemma.* Let  $u(t) = \sin t \cos t (\cos t + \sin t)$ . Then,  $u$  is an increasing function on  $[0, \frac{\pi}{4}]$  with  $u(0) = 0$  and  $u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ .

*Proof.*  $u(0) = 0, u(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$  are immediate. A simple calculation gives the derivative of  $u$ :

$$u'(t) = (\cos t - \sin t)((\cos t + \sin t)^2 + \sin t \cos t).$$

For  $t \in (0, \frac{\pi}{4})$ ,  $\cos t > \sin t$ , hence  $u'(t) > 0$  and so  $u$  is increasing on  $[0, \frac{\pi}{4}]$ .  $\square$

a) Let  $f(t) = 4^{\cos t} + 4^{\sin t}$ . We show that  $f$  is increasing on  $[0, \frac{\pi}{4}]$  (the required result then follows since  $f(0) = 5$ ). To this end, we prove that  $f'(t) > 0$  for all  $t \in (0, \frac{\pi}{4})$ . We calculate

$$f'(t) = (\ln 4)4^{\cos t} \cos t \left( 4^{\sin t - \cos t} - \frac{\sin t}{\cos t} \right)$$

so that it is sufficient to prove that  $\phi(t) > 0$  for  $t \in (0, \frac{\pi}{4})$  where

$$\phi(t) = (\sin t - \cos t)(\ln 4) - \ln(\sin t) + \ln(\cos t).$$

Now, we easily obtain  $\phi'(t) = \frac{(\ln 4)u(t)-1}{\sin t \cos t}$  with, from the lemma,

$$(\ln 4)u(t) - 1 < \frac{\sqrt{2} \ln 4}{2} - 1 < 0.$$

Therefore,  $\phi'(t) < 0$  for  $t \in (0, \frac{\pi}{4})$  and  $\phi(t) > \phi(\frac{\pi}{4}) = 0$ , as desired.

b) Similarly, we introduce  $g(t) = 6^{\cos t} + 6^{\sin t}$  whose derivative has the same sign as  $\psi(t) = (\sin t - \cos t)(\ln 6) - \ln(\sin t) + \ln(\cos t)$ . Here,

$$\psi'(t) = \frac{\ln 6}{\sin t \cos t} \cdot \left( u(t) - \frac{1}{\ln 6} \right),$$

and since  $0 < \frac{1}{\ln 6} < \frac{\sqrt{2}}{2}$ ,  $u(t) - \frac{1}{\ln 6}$  (and so  $\psi'(t)$ ) vanishes at a unique  $t_0$  in  $(0, \frac{\pi}{4})$ . From the lemma, we deduce that  $\psi'(t) < 0$  if  $0 < t < t_0$  and  $\psi'(t) > 0$  if  $t_0 < t < \frac{\pi}{4}$ . Thus,  $\psi$  is decreasing on  $(0, t_0]$  and increasing on  $[t_0, \frac{\pi}{4})$ . Since  $\psi(\frac{\pi}{4}) = 0$ , we must have  $\psi(t_0) < 0$ , and since  $\lim_{t \rightarrow 0^+} \psi(t) = \infty$ , we deduce that for some  $\alpha \in (0, t_0)$ , we have  $\psi(t) > 0$  if  $t \in (0, \alpha)$ ,  $\psi(\alpha) = 0$  and  $\psi(t) < 0$  if  $t \in (\alpha, \frac{\pi}{4})$ . Thus,  $g'(t) > 0$  if  $t \in (0, \alpha)$  and  $g'(t) < 0$  if  $t \in (\alpha, \frac{\pi}{4})$  and so  $g(t) \geq (\min(g(0), g(\pi/4))) = 7$  for all  $t \in [0, \frac{\pi}{4}]$ .

*Editor's Comments.* It turns out that AM-GM is too weak to prove this inequality when used right at the beginning; the resulting right-hand-side is too small. However, one may use AM-GM in a step of the proof, as A. Stadler did, and have things work out well; the Stadler solution is an impressive use of Taylor series and clever bounds. As well, the 'general' inequality,  $a^{\cos(t)} + a^{\sin(t)} \geq a + 1$ , is not true over the required interval for every  $a > 1$ ; plotting it for  $a = 10$ , for example, shows this.

