PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **January 1, 2017***, although late solutions will also be considered until a solution is published.*

The editor thanks Rolland Gaudet, retired professor of Université de Saint-Boniface in Winnipeg, for translations of the problems.

4121. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let s be a fixed real number such that $s \geq 1$. Let a, b, c and d be non-negative numbers that satisfy $a+b+c+d = 4s$ and $ab+bc+cd+da+ac+bd = 6$. Express the minimum value of the product $abcd$ in terms of s.

4122. *Proposed by Daniel Sitaru.*

Prove that for $n \in \mathbb{N}$, the following holds

$$\left(\frac{e^n - 1}{n}\right)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1)\cdot \ldots \cdot (e^{2n}-1)}{(2n)!}.$$

4123. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ is perpendicular to the plane of the acute triangle $A'B'C'$ at its orthocentre K. Let A, B, C be the midpoints of $B'C', C'A'$ and $A'B'$, respectively. Show that $BC > KA$ and if D on ℓ satisfies $KD = \sqrt{BC^2 - KA^2}$, that the tetrahedron $ABCD$ is isosceles. (A tetrahedron is called isosceles if its opposite edges are congruent.)

4124. *Proposed by George Apostolopoulos.*

Let A_1, B_1 and C_1 be points on the sides BC, CA and AB of a triangle ABC such that

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.$$

Prove that

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2 \left(\frac{2r}{R}\right)^4,$$

where R and r are the circumradius and the inradius of ABC, respectively.

4125. *Proposed by Stephen Su and Cheng-Shyong Lee.*

Start with a triangle $A_1A_2A_3$ in the Euclidean plane and three nonzero real numbers ℓ_1, ℓ_2, ℓ_3. Define M_k and C_k to be points on the line $A_{k+1}A_{k+2}$ such that

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k \quad \text{and} \quad C_kM_{k+1}||A_kA_{k+1}, \quad k = 1,2,3$$

(with subscripts reduced modulo 3 and distances taken to be signed, so that M_k is between A_{k+1} and A_{k+2} precisely when ℓ_k is positive). Denote by R_k the point where C_kM_{k+1} intersects $C_{k+1}M_{k+2}$, $k = 1,2,3$. Show that

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{2 + \ell_1 + \ell_2 + \ell_3 - \ell_1\ell_2\ell_3}{(1+\ell_1)(1+\ell_2)(1+\ell_3)}\right)^2,$$

where square brackets denote area.

4126. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute-angled triangle. Prove that

$$\sum_{\text{cyc}} \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\sqrt{1-\tan\frac{A}{2}\tan\frac{B}{2}}} \geq \sqrt{\frac{3}{2}}.$$

4127. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

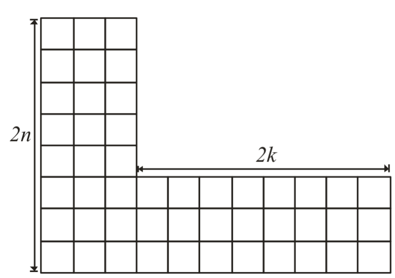
Calculate

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx,$$

where $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a continuous function.

4128. *Proposed by Valcho Milchev and Tsvetelina Karamfilova.*

Let A_n be the number of domino tilings of a rectangular $3 \times 2n$ grid. Let $L(2n, 2k)$ be the number of domino tilings of the grid composed of two rectangular grids of dimensions $3 \times 2n$ and $3 \times 2k$ with $n \geq 2$ and $k \geq 1$ (depicted below):



Prove that $L(2n, 2n) = A_{2n}$.

4129. *Proposed by Lorean Saceanu.*

Let ABC be an acute-angle triangle and let $\gamma = 3(2-\sqrt{3})$. Prove that

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$

4130. *Proposed by Leonard Giugiuc.*

Let a, b and c be nonnegative real numbers such that $a+b+c = ab+bc+ac > 0$. Prove that

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2\sqrt[n]{2}$$

for any integer $n \geq 3$ and determine the case for equality to hold.

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4121. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit s une nombre réel tel que $s \geq 1$. Soient a, b, c et d des nombres non négatifs tels que $a+b+c+d = 4s$ et $ab+bc+cd+da+ac+bd = 6$. Déterminer la valeur minimale du produit $abcd$ en terme de s.

4122. *Proposé par Daniel Sitaru.*

Démontrer que pour $n \in \mathbb{N}$, l'inégalité suivante tient

$$\left(\frac{e^n-1}{n}\right)^{2n+1} \leq \frac{(e-1)(e^2-1)(e^3-1)\cdot\ldots\cdot(e^{2n}-1)}{(2n)!}.$$

4123. *Proposé par Michel Bataille.*

Dans l'espace euclidien 3-dimensionnel, une ligne ℓ est perpendiculaire au plan du triangle aigu $A'B'C'$, à son orthocentre K. Soient A, B et C les mi points de $B'C', C'A'$ et $A'B'$, respectivement. Démontrer que $BC > KA$ et que si D sur ℓ satisfait $KD = \sqrt{BC^2 - KA^2}$, alors le tétraèdre $ABCD$ est isocèle. (Un tétraèdre est dit isocèle si ses côtés opposés sont congrus.)

4124. *Proposé par George Apostolopoulos.*

Soient A_1, B_1 et C_1 des points sur les côtés BC, CA et AB du triangle ABC, tels que

$$\frac{A_1B}{A_1C} = \frac{B_1C}{B_1A} = \frac{C_1A}{C_1B} = k.$$

Démontrer que

$$\left(\frac{AA_1}{BC}\right)^2 + \left(\frac{BB_1}{CA}\right)^2 + \left(\frac{CC_1}{AB}\right)^2 \geq \left(\frac{3k}{k^2+1}\right)^2\left(\frac{2r}{R}\right)^4,$$

où R et r sont, respectivement, les rayons des cercles circonscrit et inscrit du triangle ABC.

4125. *Proposé par Stephen Su et Cheng-Shyong Lee.*

Soient un triangle $A_1A_2A_3$ dans le plan euclidien et trois nombres réels non nuls ℓ_1, ℓ_2, ℓ_3. Définissons M_k et C_k comme étant les points sur la ligne $A_{k+1}A_{k+2}$ tels que

$$\frac{A_{k+1}M_k}{M_kA_{k+2}} = \ell_k \quad \text{and} \quad C_kM_{k+1}||A_kA_{k+1}, \quad k = 1, 2, 3$$

(les indices étant réduits modulo 3 et les distances comportant un signe, de façon à ce que M_k se trouve entre A_{k+1} et A_{k+2} précisément lorsque ℓ_k est positif). Dénotons par R_k le point d'intersection de C_kM_{k+1} et $C_{k+1}M_{k+2}$, $k = 1, 2, 3$. Démontrer que

$$\frac{[R_1R_2R_3]}{[A_1A_2A_3]} = \left(\frac{2 + \ell_1 + \ell_2 + \ell_3 - \ell_1\ell_2\ell_3}{(1 + \ell_1)(1 + \ell_2)(1 + \ell_3)}\right)^2,$$

où les crochets dénotent une surface.

4126. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle aigu. Démontrer que

$$\sum_{\text{cyc}} \frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\sqrt{1 - \tan\frac{A}{2}\tan\frac{B}{2}}} \geq \sqrt{\frac{3}{2}}.$$

4127. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

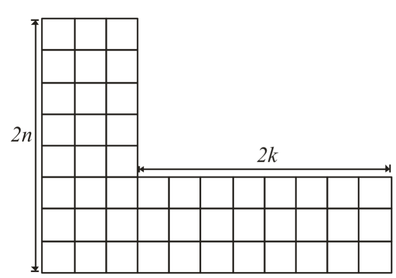
Calculer

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} f\left(\frac{x}{n}\right) dx,$$

où $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ est une fonction continue.

4128. *Proposé par Valcho Milchev et Tsvetelina Karamfilova.*

Soit A_n le nombre de carrelages par dominos d'une grille $3 \times 2n$. Soit $L(2n, 2k)$ le nombre de carrelages par dominos d'une grille formée de deux grilles rectangulaires de tailles $3 \times 2n$ et $3 \times 2k$, pour $n \geq 2$ et $k \geq 1$, comme ci-bas:



Démontrer que $L(2n, 2n) = A_{2n}$.

4129. *Proposé par Lorean Saceanu.*

Soit ABC un triangle aigu et soit $\gamma = 3(2-\sqrt{3})$. Démontrer que

$$\sec A + \sec B + \sec C \geq \gamma + \tan A + \tan B + \tan C.$$

4130. *Proposé par Leonard Giugiuc.*

Soient a, b et c des nombres réels non négatifs tels que $a+b+c = ab+bc+ac > 0$. Démontrer que

$$\sqrt[n]{a} + \sqrt[n]{b} + \sqrt[n]{c} \geq 2\sqrt[n]{2}$$

pour tout entier $n \geq 3$ et déterminer tout cas où l'égalité tient.

Math Quotes

To the pure geometer the radius of curvature is an incidental characteristic - like the grin of the Cheshire cat. To the physicist it is an indispensable characteristic. It would be going too far to say that to the physicist the cat is merely incidental to the grin. Physics is concerned with interrelatedness such as the interrelatedness of cats and grins. In this case the "cat without a grin" and the "grin without a cat" are equally set aside as purely mathematical phantasies.

Sir Arthur Eddington, "The Expanding Universe." Cambridge University Press, 1988.