FOCUS ON...

No. 21

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The Product of Two Reflections in the Plane

Introduction

In this number, we exploit the two following theorems as a tool for geometrical properties or problems. $\mathbf{R}_n$ denotes the reflection in the line n.

(a) If the lines ℓ and m are parallel, then $\mathbf{R}_m \circ \mathbf{R}_\ell = \mathbf{T}_{2\overrightarrow{u}}$, the translation by vector $2\overrightarrow{u}$ where $\overrightarrow{u}$ is the vector orthogonal to ℓ and m such that $\mathbf{T}_{\overrightarrow{u}}(\ell) = m$.

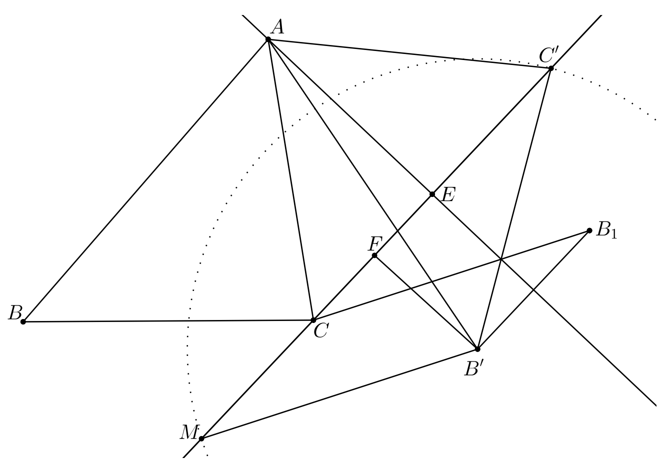
(b) If the lines ℓ and m intersect in O, then $\mathbf{R}_m \circ \mathbf{R}_\ell = \boldsymbol{\rho}_{O,2\theta}$, the rotation with centre O and angle 2θ where θ is the angle such that $\boldsymbol{\rho}_{O,\theta}(\ell) = m$.

Illustrating (a) and (b)

Our first problem offers an opportunity of using both (a) and (b):

> Let ABC be a triangle and $\boldsymbol{\rho}$ be any rotation with centre A. Let $B' = \boldsymbol{\rho}(B), C' = \boldsymbol{\rho}(C)$, and $B_1 = \mathbf{R}_{CA}(B)$. The circle with centre B' and radius $B'C'$ intersects CC' again at M. Show that MCB_1B' is a parallelogram.

Let E be the midpoint of CC'. From (b), $\mathbf{R}_{AE} \circ \mathbf{R}_{CA}$ is a rotation with centre A. In addition, it transforms C into $\mathbf{R}_{AE}(C) = C'$ (since $AC' = AC$, AE is the perpendicular bisector of CC'). Thus, $\mathbf{R}_{AE} \circ \mathbf{R}_{CA} = \boldsymbol{\rho}$ and so $B' = \mathbf{R}_{AE}(B_1)$.



In a similar way, if F is the midpoint of MC', we have $M = \mathbf{R}_{FB'}(C')$. Now, $\mathbf{R}_{AE} \circ \mathbf{R}_{FB'} = \mathbf{T}_{\overrightarrow{B'B_1}}$ (from (a)) and so $\mathbf{T}_{\overrightarrow{B'B_1}}(M) = \mathbf{R}_{AE}(C') = \mathbf{R}_{AE} \circ \boldsymbol{\rho}(C) = \mathbf{R}_{CA}(C) = C$. Thus $\overrightarrow{MC} = \overrightarrow{B'B_1}$ and the result follows.

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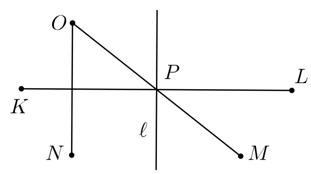
When ℓ and m are perpendicular

This is the simplest case of (b): the product $\mathbf{R}_m \circ \mathbf{R}_\ell$ is the half-turn about the point O of intersection of ℓ and m. The same is true of $\mathbf{R}_\ell \circ \mathbf{R}_m$. By way of illustration, consider the gist of problem **OC24** [2011 : 275 ; 2012 : 180]:

> Let P be the midpoint of the line segment KL and O be any point not on the line KL. Then, if M is the symmetric of O about P and N the reflection of O in KL, the points K, L, M, N are concyclic.

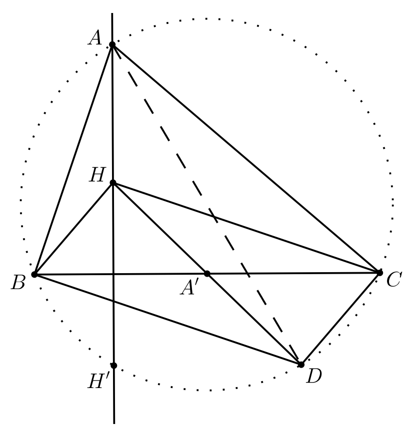
Let $\boldsymbol{\rho}_P$ be the half-turn about P. Then $N = \mathbf{R}_{KL} \circ \boldsymbol{\rho}_P(M)$ that is, $N = \mathbf{R}_\ell(M)$ where ℓ is the perpendicular to KL at P (since $\boldsymbol{\rho}_P = \mathbf{R}_{KL} \circ \mathbf{R}_\ell$). Now, let $\mathcal{C}$ be the circle through K, L, M. As the perpendicular bisector of KL, the line ℓ is a diameter of $\mathcal{C}$, hence an axis of symmetry of $\mathcal{C}$. Since M is on $\mathcal{C}$, $N = \mathbf{R}_\ell(M)$ is on $\mathcal{C}$ as well, and so K, L, M, N are concyclic.



Interestingly, this result readily leads to an unusual proof of the following well-known property of the orthocentre H of a triangle ABC: the reflections of H in the sides of the triangle lie on its circumcircle Γ.

For example, let us show that the reflection H' of H in BC is on Γ.

Let A' be the midpoint of BC. From what we have just proved, it is sufficient to show that the symmetric D of H about A' is on Γ.

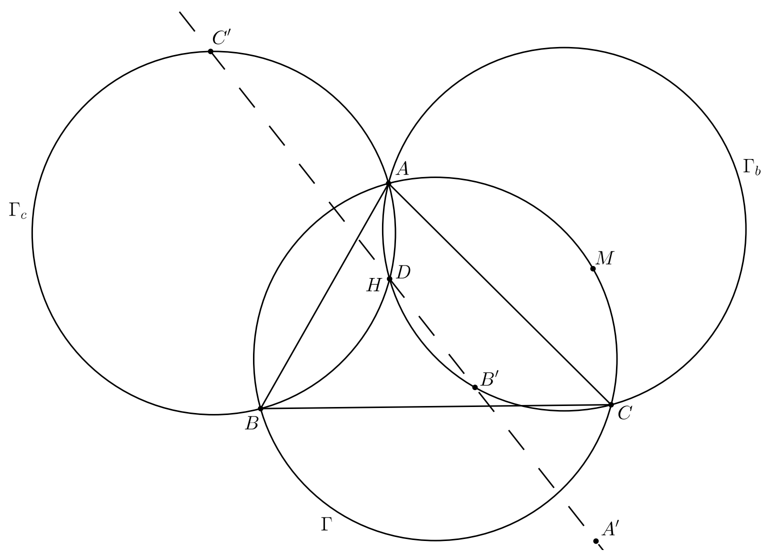


Since BC and HD have the same midpoint, $BHCD$ is a parallelogram. Thus DC is parallel to BH, hence is perpendicular to CA. Similarly, DB is perpendicular to BA and so the circle with diameter AD, which passes through A, B, and C, coincides with Γ. Thus D is on Γ.

Another unusual proof of a well-known property

> Let M be a point of the circumcircle Γ of a triangle ABC and let $A' = \mathbf{R}_{BC}(M)$, $B' = \mathbf{R}_{CA}(M)$, $C' = \mathbf{R}_{AB}(M)$. Then A', B', C' are collinear on a line through the orthocentre H (the Steiner line associated with M).

We discard the obvious case when M is a vertex of the triangle and assume that $M \neq A, B, C$.



Let Γ_b and Γ_c be the circumcircles of $\Delta AB'C$ and $\Delta AC'B$, respectively. Then $\Gamma_b = \mathbf{R}_{CA}(\Gamma)$ (since Γ is also the circumcircle of ΔAMC) and, similarly, $\Gamma_c = \mathbf{R}_{AB}(\Gamma)$. Let D be the point of intersection other than A of Γ_b and Γ_c. The product $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}$ is a rotation with centre A transforming Γ_b into Γ_c. A rotation being a spiral similarity, it follows from the main result of Focus On... No 12 that $\mathbf{R}_{AB}(C) = \mathbf{R}_{AB} \circ \mathbf{R}_{CA}(C)$ is the point of intersection other than D of the line CD and Γ_c. Thus, CD is perpendicular to AB. In the same way, BD is perpendicular to AC and consequently D coincides with the orthocentre H of ΔABC. To conclude, it suffices to add that since $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(B') = \mathbf{R}_{AB}(M) = C'$, the points B', C', H are collinear. The same being true of A', B', H and of C', A', H (similarly), we conclude that A', B', C', H are collinear.

Reflections and billiards

A billiards table is a good place for observing reflections. Successive reflections in the cushions are sometimes necessary to score a point! The following problem (a part of *American Mathematical Monthly* problem 10749 posed in 1999) is likely to appeal to the reader who is also a billiards player.

> Let ABC be a triangle with a right angle at B and an angle of $\pi/6$ at

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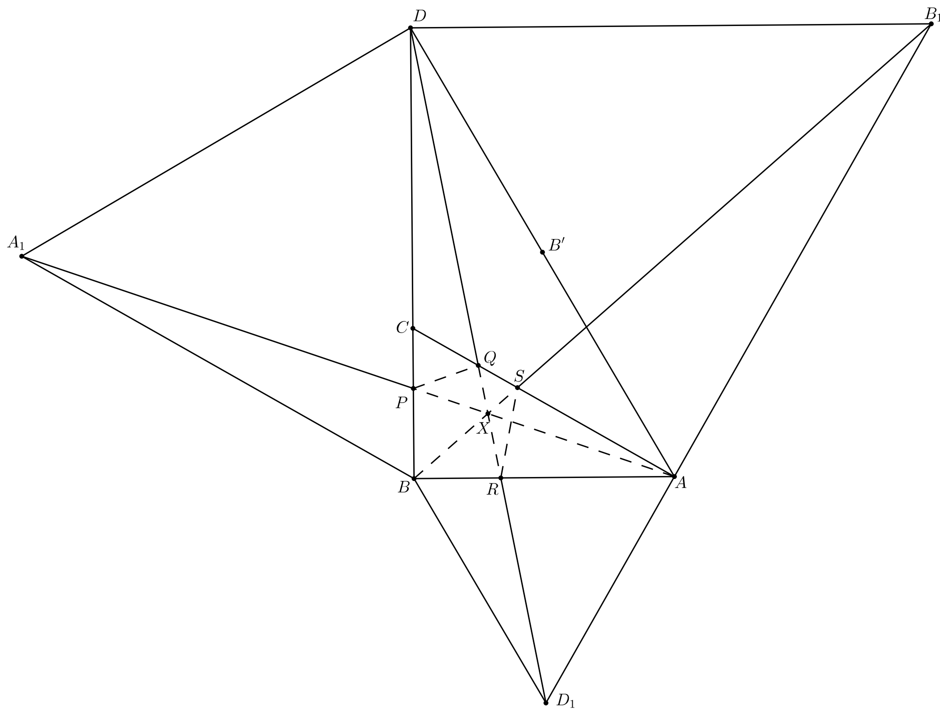
A. Consider a billiard path in the triangle that begins at A, reflects successively off side BC at P, off side AC at Q, off side AB at R, off side AC at S, and then ends at B. Show that AP, QR, and SB are concurrent at a point X and that the angles formed at X measure $\pi/3$.



Without loss of generality, we suppose that the triangle ACB has positive orientation and introduce the following points: $B' = \mathbf{R}_{CA}(B)$, the point D of intersection of the lines BC and AB', and the vertices A_1, B_1, D_1 of equilateral triangles A_1BD, B_1DA, D_1AB constructed outward ΔABD. From the billiard path as described, we have $\mathbf{R}_{BC}(AP) = PQ$, $\mathbf{R}_{CA}(PQ) = QR$, $\mathbf{R}_{AB}(QR) = RS$, and $\mathbf{R}_{CA}(RS) = SB$ so that

$$SB = (\mathbf{R}_{CA} \circ \mathbf{R}_{AB} \circ \mathbf{R}_{CA}) \circ \mathbf{R}_{BC}(AP) = \mathbf{R}_{AB'} \circ \mathbf{R}_{BC}(AP) \qquad (1)$$

(using the easily checked fact that $\mathbf{R}_m \circ \mathbf{R}_\ell \circ \mathbf{R}_m$ is the reflection in the line $\mathbf{R}_m(\ell)$).

Since $\mathbf{R}_{AB'} \circ \mathbf{R}_{BC}$ is the rotation $\boldsymbol{\rho}_{D,\pi/3}$, the point $B_1 = \boldsymbol{\rho}_{D,\pi/3}(A)$ is on the line SB. Likewise, from $AP = \boldsymbol{\rho}_{D,-\pi/3}(SB)$, we obtain that A_1 is on AP. Furthermore, (1) yields $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(SB) = \mathbf{R}_{CA} \circ \mathbf{R}_{BC}(AP)$, that is, $QR = \boldsymbol{\rho}_{C,2\pi/3}(AP)$ and therefore the points $\boldsymbol{\rho}_{C,2\pi/3}(A) = D$ and $\mathbf{R}_{AB} \circ \mathbf{R}_{CA}(B) = \boldsymbol{\rho}_{A,\pi/3}(B) = D_1$ lie on the line QR. In conclusion, the lines AP, QR, and SB coincide with the lines AA_1, DD_1, and BB_1, respectively. This answers the questions since these lines are concurrent at the Fermat point (say X) of ΔABD, from which each side subtends an angle of $120°$ (a well-known result).

Exercises

Our first exercise is problem **2439** [1999 : 238 ; 2000 : 241]. Three solutions were featured and the reader is asked to find a fourth one! The second exercise is problem **2485** [1999 : 431 ; 2000 : 508], slightly modified. Of course, solutions should use reflections.

1. Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD, respectively, such that $\angle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD, respectively. Prove that $AE + AF \geq 2\sqrt{2}a$. [Hint: first show that $\mathbf{R}_{AP}(B) = \mathbf{R}_{AQ}(D)$.]

2. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$ and such that AD and BC are not parallel. Let P be the intersection of the diagonals AC and BD. If $AP : BD = DP : AC$, prove that $AB \perp CD$. [Hint: if ℓ, m, n are the perpendicular bisectors of BC, CA, BD, respectively, and O is the circumcentre of ΔBPC, consider $\mathbf{R}_{OC} \circ \mathbf{R}_\ell$ and $\mathbf{R}_n \circ \mathbf{R}_m$.]

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