The use of coordinate systems before Descartes

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1 Introduction

It is basic knowledge among mathematicians that René Descartes introduced rectangular coordinate systems in the 17th century, thus creating analytic geometry. This accomplishment was an early example of how a branch of mathematics, algebra, can explore another area, geometry. The division of mathematics into various fields was in its early stages at that time, so this way of looking at Descartes's achievement is a rather contemporary point of view, which reflects our current (though perpetually changing) classification of mathematical subjects.

Descartes published his ideas on this topic in 1637 in the appendix *La géométrie* (see Figure 1) to his book *Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences* (Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences), a philosophical and autobiographical treatise that had a strong influence on the further development of human thought. The concept of *Cartesian coordinate system* was coined to honour Descartes, whose Latanized name was Cartesius. Students of mathematics who have struggled to solve geometry problems using the ancient synthetic methods can get a good feeling about the power of this approach, the more so if they can surmount the computational hurdles that occur sometimes. Apart from this merit, Descartes's contribution opened the way towards laying bridges between branches of mathematics that had developed separately, a feat that is highly regarded among researchers today because it emphasizes the unity of our field of knowledge, in spite of the often overlapping areas in which we divide it.

Some critics, however, contested Descartes's priority for designing this method, since Pierre Fermat had the same idea a few years before Descartes. A quote from E.T. Bell, an American mathematician and colourful historian of mathematics, sheds more light on this issue, [1]: "There is no doubt that he [Fermat] preceded Descartes. But as his work of about 1629 was not communicated to others until 1636, and was published posthumously only in 1679, it could not possibly have influenced Descartes in his own invention, and Fermat never hinted that it had." Since similar simultaneous independent contributions to science and mathematics are common, a phenomenon that has been thoroughly researched, we will not pursue this topic.

What we actually want to emphasize here is a little known fact, namely that hidden rectangular coordinate systems have been used in geometry since antiquity. We will further present two examples of how this idea was applied long before Descartes made it universally known. The first is due to the Greek geometer and astronomer

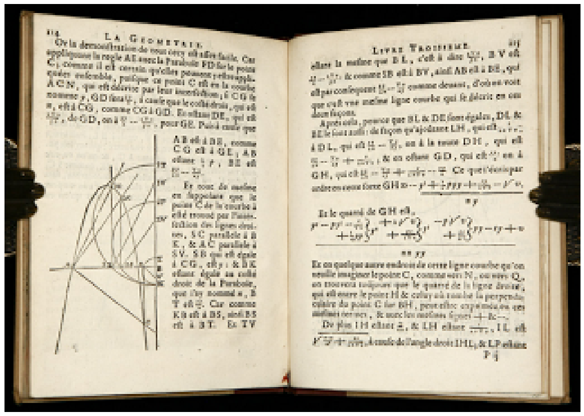

Figure 1: An extract from *La géométrie*, the appendix of *Discourse de la méthode* in which Cartesian coordinates were introduced.

Apollonius of Perga (c. 262–c. 190 BC), whereas the second is drawn from the work of the Persian mathematician Umar ibn Ibrahim al-Khayyami (AD 1048–1131), known among literati as the poet Omar Khayyam.

2 The conics of Apollonius

Apollonius was born in Perga, Pamphylia, today Murtina in Antalya, Turkey, around 262 BC. He studied in Alexandria under disciples of Euclid, where he learned geometry and astronomy, subjects he later taught there. We know very little about his life. The information we can trust appears in the various prefaces of his fundamental treatise, Κωνικά (Conics), whose original was lost (as it happened with most ancient manuscripts), but which reached us mostly through Arabic translations.

It is intuitively easy to understand why the ellipse, parabola, and hyperbola are called conics: if we intersect a cone of two sheets with a plane at suitable angles relative to the symmetry axis, we obtain these curves (along with circles, points, and two intersecting straight lines). Apollonius was the first to coin their names, though other Greek mathematicians studied them earlier. In his treatise, he approached them without using algebra, as we define this branch of mathematics today. So how did he do it? Unlike polygons or circles, conics are difficult to understand without analytic geometry, which describes them through simple quadratic equations. A close look at the work of Apollonius, however, reveals that he used camouflaged coordinate systems, and in the absence of algebraic tools he employed a geometric language. So, with his smart and systematic approach in the study of conics, he anticipated the work of Descartes by almost two millennia.

As an example, we consider now the case of the parabola (see Figure 2), but let us remark that Apollonius used this method to understand hyperbolas and ellipses too (see, e.g., [2]). Let us take a circular cone (Figure 2, left), which we intersect with a plane parallel with the generatrix AC. The result of this intersection is what we call a parabola, the curve FNH. Apollonius wanted to characterize this curve in order to distinguish it from ellipses and hyperbolas. For this purpose he derived the "symptom," which in our modern language is nothing but the relationship between the abscissa and the ordinate of an arbitrary point on the curve. So let M be such a point. A plane through M orthogonal to the symmetry axis intersects the cone along the circle of diameter DE. Denote by O the point at the base of the perpendicular from M to DE. Then ON and OM are also perpendicular to each other. Apollonius wanted to find the relationship between $ON =: x$ and $OM =: y$. Indeed, had he placed a coordinate system with the origin at N in the plane of the parabola (Figure 2, right), M would have had coordinates (x, y), the standard notation we use today to write the equation of the parabola.

Next, Apollonius started his geometric reasoning. He noticed that in the circle of diameter DE he could write that

$$OM^2 = OD \cdot OE. \tag{1}$$

Then he constructed the segment NK perpendicular to ON in the plane of the parabola (Figure 2, right), taking its length such that

$$\frac{NK}{NA} = \frac{BC}{AC} \cdot \frac{BC}{AB}, \tag{2}$$

a brilliant choice he probably reached after long reflections on this problem. By similarity of triangles, he noticed on one hand that

$$\frac{BC}{AC} = \frac{OD}{ON}, \tag{3}$$

and on the other hand, after manipulating some proportions, that

$$\frac{BC}{AB} = \frac{OE}{NA}. \tag{4}$$

Substituting the results from (3) and (4) into (2), he obtained that

$$\frac{NK}{NA} = \frac{OE \cdot OD}{NA \cdot ON}.$$

Multiplying NK/NA by ON/ON, i.e. by 1, he got

$$\frac{NK}{NA} = \frac{NK \cdot ON}{NA \cdot ON}.$$

Comparing the numerators of the above two relationships, he concluded that

$$OE \cdot OD = NK \cdot ON,$$

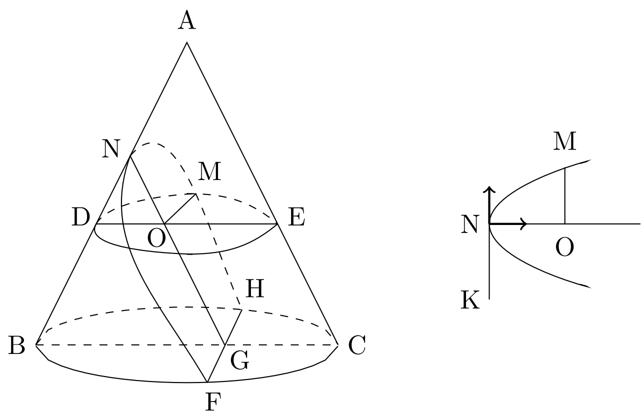

Figure 2: Left: the intersection of the cone with a plane parallel with the generatrix AC yields a parabola. Right: the parabola FNH viewed in its own plane.

which in light of (1) can be written as

$$OM^2 = NK \cdot ON.$$

If we denote the length of NK by p, which is the same as the length of the latus rectum (defined below), then the above relationship becomes

$$y^2 = px,$$

easy to recognize as the equation of a parabola, albeit written in nonstandard form.

So in spite of having no algebra tools at his disposal, Apollonius actually obtained the equation of the parabola. He even had a value for the length of the latus rectum, defined as the chord through the focus of the parabola parallel with the directrix. All in all, he had a very good understanding of conics, which he achieved through the same kind of analysis Descartes's did, but using a very basic language.

3 The cubic equation of al-Khayyami

The mathematician, philosopher, and poet al-Khayyami was born in AD 1048 in Nishapur, Persia, today a city of the same name in northeastern Iran. During his long life he contributed to mathematics, mechanics, astronomy, geography, mineralogy, and philosophy, and left an important body of poetry, which is still widely read. Five centuries before Cardano's formula for the cubic equation was discovered, al-Khayyami approached the problem with innovative methods. Like Apollonius, he used hidden Cartesian coordinates to express his solutions.

At that time all coefficients of an equation were considered positive. (Although negative numbers had been used in China no later than the AD 100, see [2],

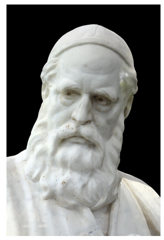

Figure 3: A bust of al-Khayyami in Nishapur, Iran.

p. 198, they became universally accepted only after the 16th century.) So al-Khayyami first listed all possible cubic equations: one binomial, $x^3 = d$; six trinomial, $x^3 + bx^2 = d$, $x^3 + d = bx^2$, $x^3 = bx^2 + d$, $x^3 + cx = d$, $x^3 + d = cx$, $x^3 = cx + d$; and seven tetranomial, $x^3 + bx^2 + cx = d$, $x^3 + bx^2 + d = cx$, $x^3 + cx + d = bx^2$, $x^3 = bx^2 + cx + d$, $x^3 + bx^2 = cx + d$, $x^3 + cx = bx^2 + d$, $x^3 + d = bx^2 + cx$. Then he provided for each of them a solution obtained with the help of a conic section, as we will show in the following example. Let us consider the case of

$$x^3 + cx = d. \tag{5}$$

He first took a segment $AB = \sqrt{c}$ and drew $EB = d/c$ perpendicular to AB (see Figure 4). Then he considered the parabola

$$y = \frac{1}{\sqrt{c}}x^2$$

through D and B, and constructed the semicircle EDB given by

$$\left(x - \frac{d}{2c}\right)^2 + y^2 = \frac{d^2}{4c^2}. \tag{6}$$

The semicircle and the parabola intersect at D. Taking $DF = y_0$ perpendicular to EB, he obtained the segment $FB = x_0$ and claimed that its length, x_0, is a solution of equation (5). Notice the hidden upside-down frame centred at B that occurs here, which allowed us to write the above equations in modern language.

Following [2], let us now prove that x_0 is indeed a solution of equation (5). Since D is on the semicircle, we have

$$x_0\left(\frac{d}{c} - x_0\right) = y_0^2,$$

an equation equivalent to (6) that can be also expressed as

$$\frac{x_0}{y_0} = \frac{y_0}{\frac{d}{c} - x_0}. \tag{7}$$

But D is also on the parabola, so we have

$$x_0^2 = \sqrt{c}y_0,$$

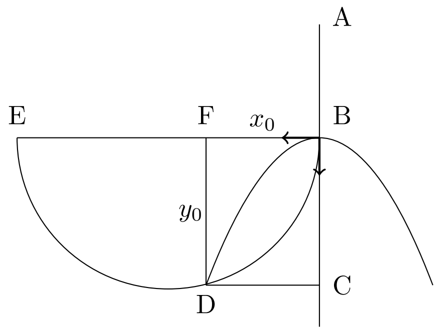

Figure 4: A solution al-Khayyami gave for the equation he described as "a cube and sides are equal to a number," i.e. $x^3 + ax = b$.

which is equivalent to

$$\frac{\sqrt{c}}{x_0} = \frac{x_0}{y_0}.$$

Squaring this equality and using (7), we obtain

$$\frac{c}{x_0^2} = \frac{x_0^2}{y_0^2} = \frac{y_0^2}{\left(\frac{d}{c} - x_0^2\right)^2} = \frac{y_0}{\frac{d}{c} - x_0} \cdot \frac{x_0}{y_0} = \frac{x_0}{\frac{d}{c} - x_0}.$$

But comparing the first and last expressions in the above sequence, we see that x_0 verifies equation (5), a remark that completes the argument.

Of course, al-Khayyami did not write his proof as presented here. He used only words and figures, as it can be seen in one of his manuscripts (Figure 5). Even equation (5) appeared without symbols. He described it as: "a cube and sides are equal to a number." Nevertheless, he did algebra, a branch of mathematics that had recently formed. As in the case of Apollonius, coordinate systems were essential for obtaining these results.

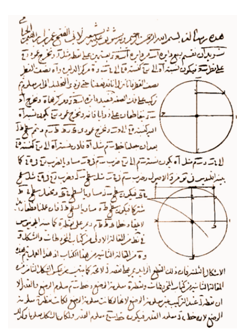

Figure 5: A page of al-Khayyami's manuscript "Cubic equation and intersection of conic sections," kept at the University of Tehran, Iran.

4 Conclusions

The modern language of mathematics is fairly recent. Even Isaac Newton's *Principia*, originally published in 1687, used geometry to express derivatives and integrals. The advantage of introducing specialized notation is tremendous for progress in research, but it also makes understanding difficult for the uninitiated. Since plain language and figures was all that mathematicians used in antiquity and the Middle Ages, the development of the field was very slow. It started booming only after the 18th century, when modern symbolism allowed a better understanding of mathematical objects and of the relationships that govern them.

Nevertheless, the ancients already had some of the ideas that later crystallized and formed new branches of mathematics. Apart from the coordinate system mentioned above, a well-known example is that of the integral, which lies at the foundation of calculus. Some two millennia before the birth of mathematical analysis, Archimedes approximated the length of a circle with regular polygons of many sides, a procedure that leads to an integral in the limit. In a way, these examples resemble the wheel, which was invented in prehistoric times. We added ball bearings and tires, improved and extended their use, but wheels are still based on the same idea. So we should not be too surprised that the ancient Greeks employed the rectangular coordinate system about 2,200 years ago, long before Fermat and Descartes understood its value.

References

[1] E.T. Bell, *Development of Mathematics*, 2nd. ed., McGraw-Hill, New York, 1945.

[2] V.J. Katz, *A History of Mathematics*, 3rd. ed., Addison-Wesley, New York, 2009.

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