CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2015: 41(3), p. 96–97.

CC161. A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

A reformulation of #4 of the Santa Clara University High School Mathematics 2001 Contest.

We received ten solutions, of which eight were complete and correct. All eight solutions were nearly identical so we present a composite solution here.

We have that $n = 211$ in base b. This requires $b > 2$ and means

$$n = (211)_b = 2 \cdot b^2 + 1 \cdot b + 1,$$

while $n = 110$ in base $b+2$ gives us

$$n = (110)_{b+2} = 1 \cdot (b+2)^2 + 1 \cdot (b+2) + 0.$$

Equating these two expressions gives

$$\begin{aligned} 2b^2 + b + 1 &= b^2 + 4 + 4b + b + 2 \\ b^2 - 4b - 5 &= 0 \\ b &= -1, 5 \end{aligned}$$

We discard the negative solution both because of the restriction on b and the fact that a base cannot be negative. Using $b = 5$ we can calculate

$$n = (211)_5 = (110)_7 = 56.$$

Therefore $n = 56$ and $b = 5$ in base 10.

CC162. What is the probability that 99 divides a randomly chosen 4-digit palindrome?

A reformulation of #3 from the team section of the 2010 Raytheon MATHCOUNTS State Competition.

We received eight submissions of which four were correct and complete. We present the solution by Titu Zvonaru.

A 4-digit palindrome is a number of the form $\overline{abba}$, with $a = 1, 2, \ldots, 9$ and $b = 0, 1, \ldots, 9$, hence there are 90 numbers which are 4-digit palindrome. Since $\overline{abba} = 1001a + 100b = 11(91a + 10b)$, we deduce that all 4-digit palindromes are divisible

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by 11. The number $\overline{abba}$ is divisible by 9 if and only if $a+b$ is divisible by 9. If $a+b=9$ we have the possibilities $1881, 2772, 3663, 4554, 5445, 6336, 7227, 8118, 9009$; if $a+b=18$, then there is only the number 9999. The searched probability is $10/90 = 1/9$.

Editor's Comments. Some solvers counted also the case when $a=b=0$ in the solution, but there is a flaw. Indeed, they have counted a total of 90 palindrome numbers (9 possibilities for the nonzero digit a and 10 possibilities for the digit b), but then they counted the case when $a=b=0$, a contradiction. The solution could have been consistent if they also counted the degenerate case when $a=0$ in the total number of palindromes, giving $10 \cdot 10 = 100$ palindrome numbers. In this case we also have $a=0, b=0$ and $a=0, b=9$, giving the probability $12/100 = 3/25$. The only consistent (but not correct in the strict sense) solutions are the ones given by Kathleen E. Lewis and Hannes Geupel.

CC163. If x is randomly chosen in $[-100, 100]$, what is the probability that $g[f(x)]$ is negative given that $f(x) = x^2 + 3x - 7$ and $g(x) = x^2 - 2x - 99$?

A reformulation of #8 of the 2014 University of North Colorado Math Contest.

We received eight submissions of which seven were correct and complete. We present the solution by Titu Zvonaru.

Since $g(x) = (x+9)(x-11)$, we have $g(x) < 0 \iff x \in (-9, 11)$. It follows that

$$\begin{aligned} g(f(x)) < 0 &\iff -9 < f(x) < 11 \\ &\iff -9 < x^2 + 3x - 7 < 11 \\ &\iff (x+6)(x-3) < 0 \text{ and } (x+1)(x+2) > 0. \end{aligned}$$

Thus, $g(f(x)) < 0 \iff x \in (-6, -2) \cup (-1, 3)$, and the probability is given by the total length of the combined intervals divided by the total length of the domain of x, which is

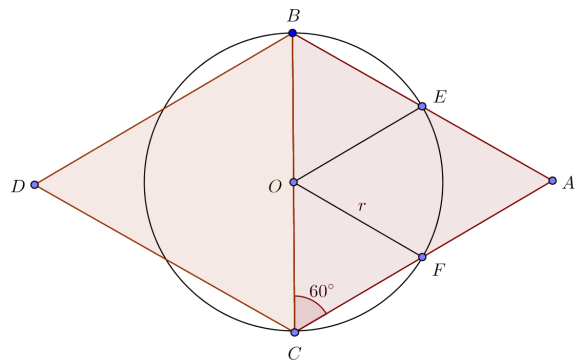
$$\frac{(-2+6) + (3+1)}{100+100} = \frac{1}{25}.$$

CC164. Build two equilateral triangles on the diameter of a circle with radius 5. What is the total area of the circle outside the equilateral triangles? (See the diagram below.)

Proposed by the editor.

We received eleven submissions of which ten were correct. We present the solution by Fernando Ballesta Yagüe, slightly modified by the editor.

Denote the center of the circle by O, the two equilateral triangles by ABC and DBC (with BC being the diameter of the circle), and the intersections of AB and AC with the circle by E and F respectively. Use r for the radius of the circle, and recall that $r=5$.



As $\triangle ABC$ is equilateral, $\angle BCA = 60°$. Further, $OC = OF = r$, so it follows that $\triangle OCF$ is also equilateral, and $\angle FOC = 60°$.

Hence the area of the circular segment between the chord CF and the circle is equal to the area of a circular sector with central angle $60°$ minus the area of the equilateral $\triangle FOC$, that is

$$\frac{\pi \cdot r^2}{6} - \frac{1}{2} \cdot r^2 \cdot \sin(60°) = \frac{25\pi}{6} - \frac{25\sqrt{3}}{4}.$$

We can reason the same way with $\triangle OBE$, and also with the matching construction on $\triangle DBC$. It follows that the area contained inside the circle but outside the triangles consists of four congruent circular segments, and the total area is

$$4 \cdot \left(\frac{25\pi}{6} - \frac{25\sqrt{3}}{4} \right) = \frac{50\pi}{3} - 25\sqrt{3}.$$

CC165. Georges pays \$50 on each of four gas refills but the prices per litre were \$1.32, \$1.25, \$1.11 and \$1.18 as the price was fluctuating a lot in that time period. What is the average price per litre?

Proposed by the editor.

We received five correct solutions and one incorrect solution. We present the solution of Henry Ricardo.

The quantities of gas purchased were $\frac{\$50}{\$1.32/L}$, $\frac{\$50}{\$1.25/L}$, $\frac{\$50}{\$1.11/L}$, and $\frac{\$50}{\$1.18/L}$.

$$\begin{aligned}
\text{(Average price per litre)} &= \frac{\text{(Total cost of gas)}}{\text{(Total quantity of gas purchased)}} \\
&= \frac{\$200}{\frac{\$50}{\$1.32/L} + \frac{\$50}{\$1.25/L} + \frac{\$50}{\$1.11/L} + \frac{\$50}{\$1.18/L}} \\
&\approx \$1.21/L
\end{aligned}$$

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