# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2014: 40(3), p. 120–124.*

## 3921. *Proposed by Michel Bataille.*

Let $AUVW$ be a rectangle with $UV = 1$. In each of the following cases (a) $VW = 8$ and (b) $VW = 9$, is it possible to construct with ruler and compass points $B$ on ray $[AU)$ and $C$ on ray $[AW)$ such that $V$ and the orthogonal projection of $A$ onto $BC$ are symmetric about the midpoint of $BC$?

*We received six correct submissions. We present the solution by Titu Zvonaru.*

With $D$ denoting the orthogonal projection of $A$ onto $BC$, the constructed figure will consist of a right triangle $ABC$ with points $V$ and $D$ on the hypotenuse $BC$ so that $BV = CD$. Denoting $a = AU$ and $x = UB$, we obtain by similitude ($\Delta ABC \sim \Delta UBV \sim \Delta DAC$),

$$AC = \frac{x+a}{x} \quad \text{and} \quad CD = \frac{AC^2}{BC}.$$

Consequently,

$$BC^2 = (x+a)^2 + \frac{(x+a)^2}{x^2} = \frac{(x+a)^2(x^2+1)}{x^2} \quad \text{and} \quad BV^2 = x^2+1.$$

Now, $BV^2 = CD^2$ then implies that

$$x^2+1 = \frac{(x+a)^4}{x^4} \cdot \frac{x^2}{(x+a)^2(x^2+1)}, \quad \text{or} \quad (x^2+1)^2 = \left(\frac{x+a}{x}\right)^2,$$

which (because all lengths must be positive) reduces to $x = \sqrt[3]{a}$.

*Case a*). If $a = 8$, then $x = 2$ and *yes*, we can construct the point $B$ with ruler and compass so that $UB = 2 = 2UV$ (and then $C$ is the point where $BV$ intersects $AW$).

*Case b*). If $a = 9$, then we *cannot* construct with ruler and compass the required points $B$ and $C$. (A segment whose length is the cube root of a rational number cannot be constructed unless it is rational, but $\sqrt[3]{9}$ is not rational.)

## 3922. *Proposed by Marcel Chiriţă.*

Let $M$ be a point inside a triangle $ABC$. Show that

$$\frac{(x+y+z)^9}{729xyz} \geq a^2b^2c^2,$$

where $MA = x, MB = y, MC = z$ and $BC = a, AC = b, AB = c$.

*We received four correct solutions and one comment. We present the solution by AN-anduud Problem Solving Group.*

Observe that

$$\left(\frac{1}{x}\overrightarrow{MA} + \frac{1}{y}\overrightarrow{MB} + \frac{1}{z}\overrightarrow{MC}\right) \cdot \left(\frac{1}{x}\overrightarrow{MA} + \frac{1}{y}\overrightarrow{MB} + \frac{1}{z}\overrightarrow{MC}\right) \geq 0.$$

Since $\overrightarrow{MA} \cdot \overrightarrow{MB} = xy(\cos\angle AMB) = (1/2)(x^2 + y^2 - c^2)$, with similar expressions for other dot products, we obtain that

$$3 + \frac{1}{xy}(x^2 + y^2 - c^2) + \frac{1}{yz}(y^2 + z^2 - a^2) + \frac{1}{zx}(z^2 + x^2 - b^2) \geq 0.$$

Multiplying by $xyz$ leads to

$$(xy + yz + zx)(x + y + z) \geq xa^2 + yb^2 + zc^2.$$

Since $x^2 + y^2 + z^2 \geq xy + yz + zx$, then $(x + y + z)^2 \geq 3(xy + yz + zx)$, and

$$\frac{(x + y + z)^3}{3} \geq (xy + yz + zx)(x + y + z) \geq xa^2 + yb^2 + zc^2 \geq 3\sqrt[3]{a^2b^2c^2xyz}.$$

Cubing this inequality yields the desired result.

*Editor's Comment.* Two of the other solvers evaluated $a$, $b$, $c$ in terms of $x$, $y$, $z$ using the Law of Cosines and reduced the problem to showing that the sum of the cosines of the three angles at $M$ was not less than $-3/2$. Michel Bataille noted that this is Problem J341 in *Mathematical Reflections* (3) 2015, whose solution appears in the same journal (4) 2015.

**3923**. *Proposed by George Apostolopoulos.*

Prove that in any triangle $ABC$,

$$\frac{\sin^3\frac{A}{2}}{\sin^3\frac{A}{2} + \cos^3\frac{A}{2}} + \frac{\sin^3\frac{B}{2}}{\sin^3\frac{B}{2} + \cos^3\frac{B}{2}} + \frac{\sin^3\frac{C}{2}}{\sin^3\frac{C}{2} + \cos^3\frac{C}{2}} \leq \frac{3R}{2(r+s)},$$

where $s, r$ and $R$ are the semiperimeter, the inradius and the circumradius, respectively, of the triangle $ABC$.

*Editor's Comments: We received three incorrect solutions (two of them were identical) all "proving" the given inequality which actually turned out to be false. One solver attempted to give a counterexample which is not valid. The counterexample below is by Michel Bataille.*

To see that the given inequality is incorrect, consider an isosceles right triangle $ABC$ with $\angle A = \frac{\pi}{2}$ and side lengths 1,1 and $\sqrt{2}$. Then using the facts that:

i) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$ and $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$,

ii) $R = \dfrac{a}{2\sin\angle A} = \dfrac{\sqrt{2}}{2}$,

iii) $s = \dfrac{2+\sqrt{2}}{2}$,

iv) $r = \dfrac{K}{s} = \dfrac{1}{2+\sqrt{2}} = \dfrac{2-\sqrt{2}}{2}$, where $K$ denotes the area of the triangle,

we compute and find that the left side of the proposed inequality is 0.6326 while the right side is 0.5303 (both rounded to 4 decimal places).

Arslanagić stated, without proof, that the given inequality would hold if the right side is replaced by $\dfrac{2.03R}{r+s}$.

## **3924**. *Proposed by Michel Bataille.*

Let $\{F_k\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for every positive integer $k$. If $m$ and $n$ are positive integers with $m$ odd and $n$ not a multiple of 3, prove that $5F_m^2 - 3$ divides $5F_{mn}^2 + 3(-1)^n$.

*We received three correct solutions and one incorrect solution.*

*Solution 1, by Oliver Geupel.*

Let

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Let $\{L_n\}$ denote the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_n + L_{n+1}$ for every nonnegative integer $n$. Binet's formulas state that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ for every $n \geq 0$. Let $m$ and $k$ be nonnegative integers with $m$ odd. By Binet's formulas and $\alpha\beta = -1$, we obtain

$$5F_k^2 + 2(-1)^k = 5\left(\frac{\alpha^k + \beta^k}{\alpha - \beta}\right)^2 + 2(-1)^k = \alpha^{2k} + \beta^{2k} = L_{2k}, \tag{1}$$

and furthermore

$$L_{2m}L_{2m(k+1)} = L_{2mk} + L_{2m(k+2)}. \tag{2}$$

Let $P(n)$ denote the assertion that for every odd natural number $m$,

$$L_{2mn} \equiv \begin{cases} (-1)^{n+1} \pmod{L_{2m} - 1} & \text{if } n \not\equiv 0 \pmod{3}, \\ 2(-1)^n \pmod{L_{2m} - 1} & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

We will show by induction on $n$ that $P(n)$ is valid for every $n \geq 0$. For the start cases, $L_0 = 2 \equiv 2 \pmod{L_{2m} - 1}$ and $L_{2m} \equiv 1 \pmod{L_{2m} - 1}$, which shows $P(0)$ and $P(1)$.

Given any $n \geq 2$, we show $P(n)$ under the hypotheses $P(n-2)$ and $P(n-1)$ in three cases.

First consider $n \equiv 0 \pmod 3$. By (2) and the induction hypothesis, we obtain

$$L_{2mn} = L_{2m}L_{2m(n-1)} - L_{2m(n-2)} \equiv 1 \cdot (-1)^n - (-1)^{n-1} \tag{3}$$
$$\equiv 2(-1)^n \pmod{L_{2m}-1}. \tag{4}$$

Next let $n \equiv 1 \pmod 3$. Then

$$L_{2mn} \equiv 1 \cdot 2(-1)^{n-1} - (-1)^{n-1} \equiv (-1)^{n+1} \pmod{L_{2m}-1}.$$

Finally consider $n \equiv 2 \pmod 3$. Then

$$L_{2mn} \equiv 1 \cdot (-1)^n - 2(-1)^{n-2} \equiv (-1)^{n+1} \pmod{L_{2m}-1}.$$

Hence $P(n)$ follows and the induction is complete.

Suppose that $m$ and $n$ are positive integers with $m$ odd and $n$ not a multiple of 3. By (1), we obtain $5F_m^2 - 3 = L_{2m} - 1$ and $5F_{mn}^2 + 3(-1)^n = L_{2mn} - (-1)^{n+1}$. By $P(n)$, we conclude that $5F_m^2 - 3$ divides $5F_{mn}^2 + 3(-1)^n$.

*Solution 2, by C.R. Pranesachar, slightly expanded by the editor.*

For all positive integers $m, n$ with $m$ odd and $n \not\equiv 0 \pmod 3$, let

$$Q_{mn} = \frac{5F_{mn}^2 + 3(-1)^n}{5F_m^2 - 3}.$$

Let $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$, such that $F_t = \frac{x^t - y^t}{\sqrt{5}}$. For any odd integer $m$,

$$\begin{aligned}
5F_m^2 - 3 &= (x^m - y^m)^2 - 3 = x^{2m} + y^{2m} - 2x^m y^m - 3 \\
&= x^{2m} + y^{2m} - 2(-1)^m - 3 \\
&= x^{2m} + y^{2m} - 1 \\
&= x^{2m} + y^{2m} - x^m y^m.
\end{aligned}$$

Furthermore

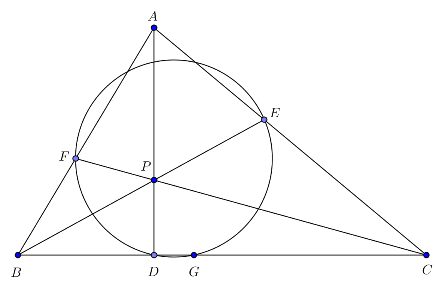
$$\begin{aligned}
5F_{mn}^2 + 3(-1)^n &= (x^{mn} - y^{mn})^2 + 3(-1)^n \\
&= x^{2mn} + y^{2mn} - 2x^{mn}y^{mn} + 3(xy)^{mn} \\
&= x^{2mn} + x^{mn}y^{mn} + y^{2mn}.
\end{aligned}$$

We know that for any positive integer $n$, the polynomial $x^{2n} + x^n + 1$ is divisible by $x^2+x+1$ if and only if $n \not\equiv 0 \pmod 3$. This is obtained by a simple application of the Remainder Theorem: if $\omega$ is a nonreal cube root of unity, then $(x-\omega)$ and $(x-\omega^2)$ are factors of $x^{2n} + x^n + 1$. Similarly, the polynomial $x^{2n} + x^n y^n + y^2 n$ is divisible by $x^2+xy+y^2$ for $n \not\equiv 0 \pmod 3$ as it has factors $(x-\omega y)$ and $(x-\omega^2 y)$ as a polynomial in $x$. Consequently, $Q_{mn}$ is a polynomial in $x$ and $y$ with integer coefficients. Since both its numerator and denominator are symmetric polynomials in $x$ and $y$, so is $Q_{mn}$. Hence $Q_{mn}$ can be expressed as a polynomial with integer coefficients in terms of $x+y$ and $xy$. But $x+y=1$ and $xy=-1$. Consequently $Q_{mn}$ is an integer, as desired.

As a remark, in the case of $n \equiv 0 \pmod 3$, the term $5F_m^2-3$ divides $F_{mn}^2$.

## **3925**. *Proposed by Ilker Can Çiçek.*

Let $ABC$ be a scalene triangle. Let $D$ be the foot of the altitude from the vertex $A$. Let $P$ be the point on the segment $AD$ ($P \neq A, P \neq D$), such that for the points $E$ and $F$ defined by $BP \cap AC = E$ and $CP \cap AB = F$, the equality $BF \cdot CD = BD \cdot CE$ holds. Let $G$ be the intersection point of the circumcircle of the triangle $DEF$ and the segment $BC$ with $G$ lying between $D$ and $C$:




Prove that $AB+AC=BC+AE$ if and only if $BF+CG=CE+BD$.

*We received three correct submissions. We present the solution by Peter Y. Woo.*

By Ceva's theorem we have

$$\frac{BF}{FA}\cdot\frac{AE}{EC}\cdot\frac{CD}{DB}=1, \tag{1}$$

so that the given equality $BF \cdot CD = BD \cdot CE$ is equivalent to

$$AE = AF.$$

We shall now see that there is exactly one possibility for the location of $P$ on $AD$: There is a unique position of $E$ on $AC$ and $F$ on $AB$ satisfying both $AE = AF$ and equation (1) when, as in our problem, the sides $b = AC$ and $c = AB$ have different lengths (and, consequently, $CD \neq DB$). Letting $x$ denote the common length of $AE = AF$ (and assuming, as the proposer intended, that the order of the points on $BC$ is $BDGC$), we see that (1) reduces to

$$x = \frac{b \cdot DB - c \cdot CD}{DB - CD},$$

which is a fixed quantity. Note also that the center of any circle through $E$ and $F$ necessarily lies on the bisector of $\angle BAC$. But we can say more: The angle bisector $AG$ is the diameter of the circumcircle of $\Delta DEF$ (see Figure 1).

To see this let $\Gamma$ be the circle whose diameter is $AG$. Then $\Gamma$ contains $D$ (the foot of the altitude from $A$). It remains to prove that $\Gamma$ intersects the sides $AC$

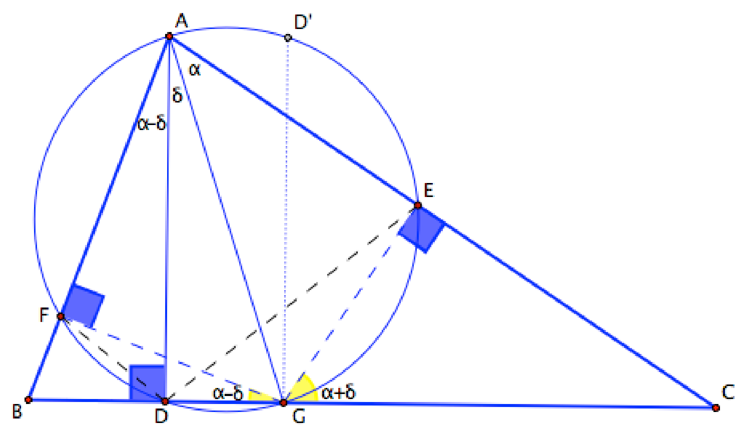



Figure 1: The angle bisector $AG$ is the diameter of the circumcircle of $\Delta DEF$.

and $AB$ at the unique points $E$ and $F$ that satisfy (1) together with $AE = AF$. Then, as in the figure, $AFG$ and $AEG$ are congruent right triangles, and we let $\alpha = \frac{1}{2}\angle BAC$ and $\delta = \angle DFG = \angle DAG = \angle DEG$. Then $\angle BFD = 90° - \delta$ and $\angle BDF = \angle FAG = \alpha$; hence (by the sine law applied to $\Delta BDF$),

$$\frac{BF}{BD} = \frac{\sin\angle BDF}{\sin\angle BFD} = \frac{\sin\alpha}{\cos\delta}.$$

Similarly, $\angle DEC = 90° + \delta$ and $\angle CDE = \angle GDE = \angle GAE = \alpha$, and

$$\frac{CE}{CD} = \frac{\sin\angle CDE}{\sin\angle DEC} = \frac{\sin\alpha}{\cos\delta}.$$

Consequently (1) holds for these positions of $E$ and $F$, and we proved the claim that these are the points on $BP$ and $CP$ described in the statement of the problem.

As in the figure, we have $\angle FGD = \angle FAD = \alpha - \delta$ and $\angle CGE = \angle DAE = \alpha + \delta$, so that

$$\frac{AB}{AD} = \frac{1}{\cos(\alpha-\delta)}, \quad \frac{AC}{AD} = \frac{1}{\cos(\alpha+\delta)},$$

$$\frac{BC}{AD} = \frac{BD + DC}{AD} = \tan(\alpha-\delta) + \tan(\alpha+\delta),$$

$$\frac{AE}{AD} = \frac{AE}{AG}\cdot\frac{AG}{AD} = \frac{\cos\alpha}{\cos\delta}.$$

The first of the given equations thereby becomes

$$\begin{aligned} 0 &= AB + AC - BC - AE \\ &= AD\left(\frac{1}{\cos(\alpha-\delta)} + \frac{1}{\cos(\alpha+\delta)} - (\tan(\alpha-\delta)+\tan(\alpha+\delta)) - \frac{\cos\alpha}{\cos\delta}\right). \end{aligned}$$

After multiplying and dividing by convenient nonzero terms this becomes, in succession,

$$\begin{aligned}
0 &= \cos(\alpha+\delta)+\cos(\alpha-\delta)-\sin 2\alpha-\frac{\cos\alpha}{\cos\delta}\cdot\frac{1}{2}(\cos 2\alpha+\cos 2\delta)\\
&= 2\cos\alpha\cos\delta-\sin 2\alpha-\frac{\cos\alpha}{2\cos\delta}(\cos 2\alpha+\cos 2\delta),\\
0 &= 4\cos\alpha\cos^2\delta-2\sin 2\alpha\cos\delta-\cos\alpha\cos 2\alpha-\cos\alpha\cos 2\delta,\\
0 &= 4\cos^2\delta-4\sin\alpha\cos\delta-\cos 2\alpha-\cos 2\delta,\\
0 &= 4\cos^2\delta-4\sin\alpha\cos\delta-1+2\sin^2\alpha-2\cos^2\delta+1,\\
0 &= \cos^2\delta-2\sin\alpha\cos\delta+\sin^2\alpha \;=\; (\cos\delta-\sin\alpha)^2.
\end{aligned}$$

By a similar calculation using the ratios

$$\frac{BF}{GF}=\tan(\alpha-\delta),\quad \frac{CE}{GF}=\frac{CE}{GE}=\tan(\alpha+\delta),\quad \frac{CG}{GF}=\frac{CE}{GE}=\frac{1}{\cos(\alpha+\delta)},$$

$$\text{and}\quad \frac{BD}{GF}=\frac{BD}{AD}\cdot\frac{AD}{AG}\cdot\frac{AG}{GF}=\tan(\alpha-\delta)\cdot\cos\delta\cdot\frac{1}{\sin\alpha},$$

the second of the given equations becomes

$$\begin{aligned}
0 &= BF-CE+CG-BD\\
&= GF\left(\tan(\alpha-\delta)-\tan(\alpha+\delta)+\frac{1}{\cos(\alpha+\delta)}-\frac{\cos\delta}{\sin\alpha}\tan(\alpha-\delta)\right),
\end{aligned}$$

whence

$$\begin{aligned}
0 &= -\sin 2\delta+\frac{1}{\sin\alpha}\left(\sin\alpha\cos(\alpha-\delta)-\cos\delta\cos(\alpha+\delta)\sin(\alpha-\delta)\right),\\
0 &= -\sin\alpha\sin 2\delta+\sin\alpha(\cos\alpha\cos\delta+\sin\alpha\sin\delta)-\cos\delta\cdot\frac{1}{2}\cdot(\sin 2\alpha-\sin 2\delta)\\
&= -\sin\alpha\sin 2\delta+\sin\alpha\cos\alpha\cos\delta+\sin^2\alpha\sin\delta-\cos\delta\sin\alpha\cos\alpha+\frac{\cos\delta\sin 2\delta}{2},\\
0 &= -\sin\alpha+\frac{\sin^2\alpha}{2\cos\delta}+\frac{\cos\delta}{2},\\
0 &= -2\sin\alpha\cos\delta+\sin^2\alpha+\cos^2\delta \;=\; (\cos\delta-\sin\alpha)^2.
\end{aligned}$$

We conclude, finally, that $AB+AC=BC+AE$ and $BF+CG=CE+BD$ are both equivalent to $\cos\delta=\sin\alpha$.

*Editor's Comments.* It seems that after all this work, we have managed to discover yet another fascinating property of the empty set: The required condition (namely, $\cos\delta=\sin\alpha$) *can never* occur (unless, of course, this editor has made an error, which *can* occur — frequently). Note that in $\Delta ADG$ we have $\cos\delta=\frac{AD}{AG}$, while in $\Delta AGE$ we have $\sin\alpha=\frac{GE}{AG}$. But equality would imply that $AD=GE$. Letting $D'$ be the point of the circle $\Gamma$ diametrically opposed to $D$, we have $AD=GD'>GE$ (because $D'$ is outside the circle with centre $G$ that meets $\Gamma$ in $E$ and $F$). In other words, there exists no triangle that satisfies all the requirements of the problem. Specifically, our assumptions (namely $AC > AB, BF \cdot CD = BD \cdot CE$, and the points along $BC$ lie in the order $BDGC$) imply that $AB + AC > BC + AE$ and $BF + CG > CE + BD$.

## **3926**. *Proposed by George Apostolopoulos.*

Let $a, b$ and $c$ be positive real numbers such that $a+b+c = 1$. Find the minimum value of the expression

$$\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}.$$

*We received twelve correct submissions. We present 3 different solutions.*

*Solution 1, by Michel Bataille.*

For $x, y > 0$, we have $2(x^2 + y^2) \geq (x + y)^2$, hence

$$x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}=(x^2+y^2)\left(1+\frac{1}{x^2y^2}\right)\geq\frac{(x+y)^2}{2}\left(1+\frac{1}{(xy)^2}\right).$$

It follows that the given expression $E$ satisfies $E \geq E'$ where

$$E'=\sqrt{2}\left(\frac{a+b}{2}\sqrt{1+\frac{1}{a^2b^2}}+\frac{b+c}{2}\sqrt{1+\frac{1}{b^2c^2}}+\frac{c+a}{2}\sqrt{1+\frac{1}{c^2a^2}}\right).$$

Consider the function $f(t) = \sqrt{1+t^2}, t > 0$. Then straightforward calculations show that

$$f'(t)=\frac{t}{\sqrt{1+t^2}}\quad\text{and}\quad f''(t)=\frac{1}{(1+t^2)^{3/2}}>0,$$

so $f$ is convex on $(0, \infty)$.

Since

$$\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}=1,$$

Jensen's inequality yields

$$E'\geq\sqrt{2}\cdot\sqrt{1+\left(\frac{a+b}{2ab}+\frac{b+c}{2bc}+\frac{c+a}{2ca}\right)^2}=\sqrt{2}\cdot\sqrt{1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}.$$

Since

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{a+b+c}=9$$

by the AM-HM inequality, we finally obtain $E \geq E' \geq \sqrt{2}\sqrt{82} \geq 2\sqrt{41}$.

It is easily checked that equality holds if and only if $a = b = c = \frac{1}{3}$.

*Solution 2, by John G. Heuver.*

By Minkowski's inequality and the AM-GM inequality, we have:

$$\begin{aligned}
&\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}\\
&\qquad\geq \sqrt{(a+b+c)^2+(b+c+a)^2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\\
&\qquad= \sqrt{2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2} \geq \sqrt{2+2\left(\frac{3}{\sqrt[3]{abc}}\right)^2}\\
&\qquad\geq \sqrt{2+2\left(\frac{9}{a+b+c}\right)^2} = \sqrt{2+2\cdot 81} = \sqrt{164} = 2\sqrt{41}.
\end{aligned}$$

*Solution 3, by Titu Zvonaru.*

Since $2(x^2+y^2) \geq (x+y)^2$, we have $\sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y)$. By the AM-HM inequality, we also have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c}.$$

Hence by using the triangle inequality for complex numbers, we obtain:

$$\begin{aligned}
&\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}\\
&\quad= \sqrt{\left|a+\frac{i}{a}\right|^2+\left|b+\frac{i}{b}\right|^2}+\sqrt{\left|b+\frac{i}{b}\right|^2+\left|c+\frac{i}{c}\right|^2}+\sqrt{\left|a+\frac{i}{a}\right|^2+\left|c+\frac{i}{c}\right|^2}\\
&\quad\geq \frac{1}{\sqrt{2}}\left(2\left|a+\frac{i}{a}\right|+2\left|b+\frac{i}{b}\right|+\left|c+\frac{i}{c}\right|\right)\\
&\quad\geq \frac{2}{\sqrt{2}}\left|(a+b+c)+i\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right| = \sqrt{2}\sqrt{(a+b+c)^2+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\\
&\quad\geq \sqrt{2}\sqrt{1+\left(\frac{9}{a+b+c}\right)^2} = \sqrt{2}\sqrt{82} = 2\sqrt{41}.
\end{aligned}$$

## **3927**. *Proposed by Marcel Chiriţă.*

Let $ABCO$ be a tetrahedron with the face angles at $O$ all right angles. If we denote the altitude from $O$ by $h$, the inradius by $r$, and the angles that the lines $OA, OB, OC$ make with the face $ABC$ by $x, y, z$, show that

$$r\sqrt{2} \geq h(\cos 2x+\cos 2y+\cos 2z).$$

*We received three correct submissions. We present the solution by Titu Zvonaru.*

The proposed inequality is false; we shall see that instead

$$r(1+\sqrt{3}) \geq h(\cos 2x + \cos 2y + \cos 2z).$$

We use square brackets to denote areas and denote by $V$ the volume of $ABCO$; letting $a = OA, b = OB$, and $c = OC$, we have

$$V = \frac{[OAB] \cdot OC}{3} = \frac{[OBC] \cdot OA}{3} = \frac{[OCA] \cdot OB}{3} = \frac{abc}{6}.$$

Because $AB^2 = a^2 + b^2,\ BC^2 = b^2 + c^2,\ CA^2 = c^2 + a^2$, we have

$$\begin{aligned} 16[ABC]^2 &= 2(a^2+b^2)(b^2+c^2) + 2(b^2+c^2)(c^2+a^2) + 2(c^2+a^2)(a^2+b^2) \\ &\qquad -(a^2+b^2)^2 - (b^2+c^2)^2 - (c^2+a^2)^2 \\ &= 4(a^2b^2 + b^2c^2 + c^2a^2). \end{aligned}$$

In terms of $h$ and $r$ the volume is

$$V = \frac{[ABC] \cdot h}{3} = \frac{r([OAB] + [OBC] + [OCA] + [ABC])}{3};$$

consequently,

$$h = \frac{abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} \quad \text{and} \quad r = \frac{abc}{ab + bc + ca + \sqrt{a^2b^2 + b^2c^2 + c^2a^2}}.$$

Let $H$ be the foot of the altitude from $O$ to the face $ABC$. Then $\sin x = \frac{OH}{OA} = \frac{h}{a}$, and

$$\sin^2 x = \frac{h^2}{a^2} = \frac{b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2},$$

so that

$$\cos 2x = 1 - 2\sin^2 x = 1 - \frac{2b^2c^2}{a^2b^2 + b^2c^2 + c^2a^2}.$$

Add this last equation to the analogous expressions for $\cos 2y$ and $\cos 2z$ to get

$$\cos 2x + \cos 2y + \cos 2z = 1.$$

We can now prove our initial claim:

$$\begin{aligned} & r(1+\sqrt{3}) \geq h(\cos 2x + \cos 2y + \cos 2z) \\ \Leftrightarrow\ & \frac{1+\sqrt{3}}{ab + bc + ca + \sqrt{a^2b^2 + b^2c^2 + c^2a^2}} \geq \frac{1}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}} \\ \Leftrightarrow\ & \sqrt{3(a^2b^2 + b^2c^2 + c^2a^2)} \geq ab + bc + ca \\ \Leftrightarrow\ & 3(a^2b^2 + b^2c^2 + c^2a^2) \geq (ab + bc + ca)^2 \\ \Leftrightarrow\ & (ab - bc)^2 + (bc - ca)^2 + (ca - ab)^2 \geq 0. \end{aligned}$$

Equality holds if and only if $a = b = c$.

*Editor's Comments.* The editors misinterpreted Chiriţă's original proposal: he intended $r$ to be the radius of the incircle of the face $\Delta ABC$ (not the inradius of the tetrahedron). Using an argument similar to that of the featured solution with his inradius, he correctly obtained the inequality of his proposal, namely $r\sqrt{2} \geq h(\cos 2x + \cos 2y + \cos 2z)$, with equality if and only if $a = b = c$.

**3928**. *Proposed by Michel Bataille.*

Let $A \in \mathcal{M}_n(\mathbb{C})$ with $\operatorname{rank}(A) \leq 1$ and complex numbers $x_1, x_2, \ldots, x_n$ such that $\sum_{k=1}^{n} x_k^2 = 1$. If

$$B = \left(\begin{array}{c|c} 0 & x_1 \cdots x_n \\ \hline x_1 & \\ \vdots & A \\ x_n & \end{array}\right)$$

and $I_{n+1}$ is the unit matrix of size $n+1$, prove that

$$\det(I_{n+1} + B) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

*We received three correct solutions. We present the solution submitted by Oliver Geupel, slightly modified by the editor.*

Since $\operatorname{rank}(A) \leq 1$, there are complex numbers $a_1, \ldots, a_n$ and $\lambda_1, \ldots, \lambda_n$ such that

$$A = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} \lambda_1 a_1 & \cdots & \lambda_1 a_n \\ \vdots & & \vdots \\ \lambda_n a_1 & \cdots & \lambda_n a_n \end{pmatrix}.$$

Denote by $I_n$ the unit matrix of size $n$. We construct two intermediate matrices $C$ and $D$, and relate the determinants of $C$, $D$ and $B$ to each other by showing how each matrix can be built up from the previous via elementary row operations. Let

$$C = \left(\begin{array}{c|c} 1 & a_1 \cdots a_n \\ \hline \lambda_1 & \\ \vdots & I_n + A \\ \lambda_n & \end{array}\right) = \begin{pmatrix} C_0 & | & \cdots & | & C_n \end{pmatrix}$$

(that is, $C_0, \ldots, C_n$ denote the columns of $C$); we obtain

$$\begin{aligned} \det C &= \det \begin{pmatrix} C_0 & | & C_1 - a_1 C_0 & | & \cdots & | & C_n - a_n C_0 \end{pmatrix} \\ &= \det \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \lambda_1 & \\ \vdots & I_n \\ \lambda_n & \end{array}\right) = 1. \end{aligned}$$

Let

$$D = \left(\begin{array}{c|c} 0 & a_1 \cdots a_n \\ \hline x_1 & \\ \vdots & I_n + A \\ x_n & \end{array}\right) = \left(\begin{array}{c|c|c} D_0 & \cdots & D_n \end{array}\right)$$

(that is, $D_0, \ldots, D_n$ denote the columns of $D$). Note that $D_i = C_i$ for $i \geq 1$. On the other hand, a simple calculation shows that

$$\begin{aligned} &D_0 - x_1 D_1 - \cdots - x_n D_n \\ &\quad = \begin{pmatrix} -(a_1x_1 + \cdots + a_nx_n) \\ -\lambda_1(a_1x_1 + \cdots + a_nx_n) \\ \vdots \\ -\lambda_n(a_1x_1 + \cdots + a_nx_n) \end{pmatrix} = -(a_1x_1 + \cdots + a_nx_n)C_0. \end{aligned}$$

Using the properties of determinants,

$$\begin{aligned} \det D &= \det \left(\begin{array}{c|c|c|c} D_0 - x_1D_1 - \cdots - x_nD_n & D_1 & \cdots & D_n \end{array}\right) \\ &= \det \left(\begin{array}{c|c|c|c} -(a_1x_1 + \cdots + a_nx_n)C_0 & C_1 & \cdots & C_n \end{array}\right) \\ &= -(a_1x_1 + \cdots + a_nx_n) \det C \\ &= -(a_1x_1 + \cdots + a_nx_n). \end{aligned}$$

Let

$$I_{n+1} + B = \left(\begin{array}{c} R_0 \\ \hline \vdots \\ \hline R_n \end{array}\right)$$

so $R_0, \ldots, R_n$ denote the rows of $I_{n+1} + B$. Note that, except for the first row, the rows of $I_{n+1} + B$ are the same as the rows of $D$. Moreover,

$$\begin{aligned} &R_0 - x_1R_1 - \cdots - x_nR_n \\ &\quad = \left( 0 \quad -a_1(\lambda_1x_1 + \cdots + \lambda_nx_n) \cdots - a_n(\lambda_1x_1 + \cdots + \lambda_nx_n) \right), \end{aligned}$$

which is $-(\lambda_1x_1 + \cdots + \lambda_nx_n)$ times the first row of $D$. Note that for the above calculation we needed to use the hypothesis that $\sum_{k=1}^{n} x_k^2 = 1$ to get that the first entry is zero. By the properties of determinants, it follows that

$$\begin{aligned}\det(I_{n+1}+B) &= \det\begin{pmatrix} R_0 - x_1R_1 - \cdots - x_nR_n \\ \hline R_1 \\ \hline \vdots \\ \hline R_n \end{pmatrix} \\ &= -\left(\lambda_1x_1 + \cdots + \lambda_nx_n\right)\det D \\ &= (\lambda_1x_1 + \cdots + \lambda_nx_n)(a_1x_1 + \cdots + a_nx_n) \\ &= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.\end{aligned}$$

This completes the proof.

**3929**. *Proposed by Péter Ivády.*

Show that for all $0 < x < \pi/2$, the following inequality holds:

$$\left(1 + \frac{1}{\sin x}\right)\left(1 + \frac{1}{\cos x}\right) \geq 5\left[1 + x^4\left(\frac{\pi}{2} - x\right)^4\right].$$

*There were eight submissions for this problem, all of which were correct. We present the solution by C.R. Pranesachar.*

We shall prove that if

$$f(x) = \left(1 + \frac{1}{\sin(x)}\right)\left(1 + \frac{1}{\cos(x)}\right), \text{ and } g(x) = 5 + x^5\left(\frac{\pi}{4} - x\right)^4, \ 0 < x < \frac{\pi}{2},$$

then

$$\min_{0<x<\frac{\pi}{2}} f(x) > 5.8 > \max_{0<x<\frac{\pi}{2}} g(x).$$

We shall give a calculus-free proof. Since $f(x)$ is symmetric about the point $x = \frac{\pi}{4}$

in $(0, \frac{\pi}{2})$, we may use the substitution $x = \frac{\pi}{4} - t$, where $-\frac{\pi}{4} < t < \frac{\pi}{4}$. Then

$$\begin{aligned}
f(x) &= \left(1 + \frac{1}{\sin(\frac{\pi}{4} - t)}\right)\left(1 + \frac{1}{\cos(\frac{\pi}{4} - t)}\right) \\
&= \frac{\left(\frac{1}{\sqrt{2}}(\cos(t) - \sin(t)) + 1\right)\left(\frac{1}{\sqrt{2}}(\cos(t) + \sin(t)) + 1\right)}{\sin(\frac{\pi}{4} - t)\cos(\frac{\pi}{4} - t)} \\
&= \frac{(\sqrt{2} + \cos(t) - \sin(t))(\sqrt{2} + \cos(t) + \sin(t))}{2\sin(\frac{\pi}{4} - t)\cos(\frac{\pi}{4} - t)} \\
&= \frac{(\sqrt{2} + \cos(t))^2 - \sin^2(t)}{\sin(\frac{\pi}{2} - t)} = \frac{2 + 2\sqrt{2}\cos(t) + \cos(2t)}{\cos(2t)} \\
&= 1 + \frac{2(\sqrt{2}\cos(t) + 1)}{2\cos^2(t) - 1} = 1 + \frac{2}{\sqrt{2}\cos(t) - 1}.
\end{aligned}$$

For $f(x)$ to be at a minimum, $\sqrt{2}\cos(t) - 1$ is at a maximum, and so $\cos(t) = 1$. This happens for $t = 0$, that is, $x = \frac{\pi}{4}$. Thus

$$\min f(x) = 1 + \frac{2}{\sqrt{2} - 1} = 3 + 2\sqrt{2} > 3 + 2(1.4) = 5.8.$$

Now, the maximum of $x(\frac{\pi}{2} - x)$ is $\frac{\pi^2}{16}$, which is attained at $x = \frac{\pi}{4}$, as

$$x\left(\frac{\pi}{2} - x\right) = \frac{\pi^2}{16} - \left(\frac{\pi}{4} - x\right)^2.$$

So

$$\max g(x) = 5 + \left(\frac{\pi^2}{16}\right)^4 = 5 + \frac{\pi^8}{16^4}.$$

Since $\pi^2 < 10$ (this follows from the fact that $\pi < 3.15$), we see that:

$$\begin{aligned}
\max g(x) < 5\left(1 + \frac{10^4}{16^4}\right) &= 5\left(1 + \frac{10^6}{16^4 \times 100}\right) \\
&= 5\left(1 + \frac{(10^3)^2}{2^16 \times 100}\right) \\
&< 5\left(1 + \frac{(2^10)^2}{2^16 \times 100}\right) \\
&= 5\left(1 + \frac{2^4}{100}\right) = 5(1 + 0.16) = 5.8.
\end{aligned}$$

Hence the inequality follows.

## **3930**. *Proposed by José Luis Díaz–Barrero.*

In a triangle $ABC$, let $a$, $b$ and $c$ denote the lengths of the sides $BC$, $CA$ and $AB$. Show that

$$\sqrt{\frac{a\,\sin^{1/2}B}{4a + b + c}} + \sqrt{\frac{b\,\sin^{1/2}C}{a + 4b + c}} + \sqrt{\frac{c\,\sin^{1/2}A}{a + b + 4c}} \le \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$

*We received nine correct submissions. We present the solution by Cao Minh Quang.*

Let

$$F = \sum_{\text{cyc}} \sqrt{\frac{a \sin^{1/2} B}{4a+b+c}}.$$

Then by Cauchy-Schwarz inequality, we have

$$F^2 \leq \sum_{\text{cyc}} \frac{a}{4a+b+c} \cdot \sum_{\text{cyc}} \sin^{1/2} A. \tag{1}$$

Since $(x+y)^2 \geq 4xy$, we have $\frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$ for all $x>0, y>0$. Hence,

$$\begin{aligned}\sum_{\text{cyc}} \frac{a}{4a+b+c} &= \sum_{\text{cyc}} \frac{a}{3a+(a+b+c)} \\ &\leq \frac{1}{4}\sum_{\text{cyc}} a\left(\frac{1}{3a}+\frac{1}{a+b+c}\right) = \frac{1}{4}\sum_{\text{cyc}}\left(\frac{1}{3}+\frac{a}{a+b+c}\right) = \frac{1}{2}. \end{aligned} \tag{2}$$

Since it is well known that $\sum_{\text{cyc}} \sin A \leq \frac{3\sqrt{3}}{2}$ we have, by using Cauchy-Schwarz inequality again, that

$$\sum_{\text{cyc}} \sin^{1/2} A \leq \left(\sum_{\text{cyc}} 1\right)^{1/2} \left(\sum_{\text{cyc}} \sin A\right)^{1/2} \leq \sqrt{3}\sqrt{\frac{3\sqrt{3}}{2}} = \sqrt{\frac{9\sqrt{3}}{2}}. \tag{3}$$

From (1)–(3), we obtain that

$$F^2 \leq \frac{1}{2}\sqrt{\frac{9\sqrt{3}}{2}} = \sqrt{\frac{9\sqrt{3}}{8}},$$

therefore,

$$F \leq \sqrt[4]{\frac{9\sqrt{3}}{8}} = \sqrt[8]{\frac{3^5}{2^6}} = \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$