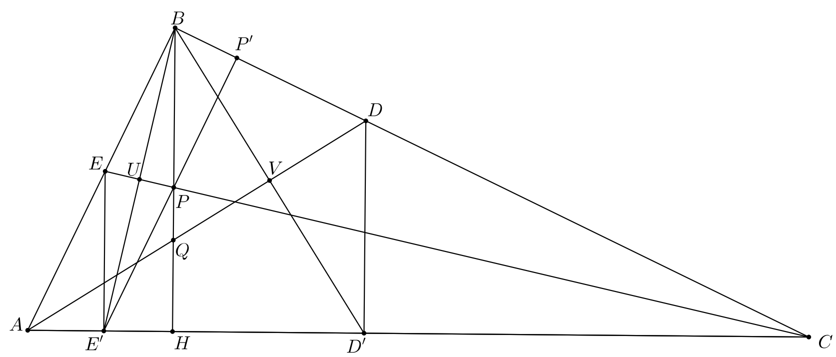
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(3), p. 102–103.

OC161. The altitude BH dropped onto the hypotenuse AC of a triangle ABC intersects the bisectors AD and CE at Q respectively P. Prove that the line passing through the midpoints of segments $[QD]$ and $[PE]$ is parallel to the line AC.

Originally problem 1 from the 2012 Grade IX Romania Math Olympiad

We received ten correct solutions. We present the solution by Michel Bataille.



Let E', D' be the orthogonal projections of E, D onto BC and let P' be the orthogonal projection of P onto BC. The lines CB and CA are symmetric in the bisector CE and it is also the case of PP' and PH (since $P \in CE$ and $PP' \perp CB, PH \perp CA$). It follows that P' and H are symmetric in CE. In a similar way, B and E' are symmetric in CE. We deduce $BP' = HE'$ and consequently

$$PB = \sqrt{PP'^2 + P'B^2} = \sqrt{PH^2 + HE'^2} = PE'.$$

Since we also have $EB = EE'$, the line CE, which passes through P and E, is the perpendicular bisector of the line segment BE'. It follows that CE intersects BE' at its midpoint U. Under the symmetry about U, the image of B is E' and the image of the line BH is the line EE' (since $EE' \parallel BH$). As a result, the image of $P = CE \cap BH$ is $CE \cap EE' = E$, that is, U is also the midpoint of PE. Similarly, the midpoint V of BD' is the midpoint of QD and the desired result follows since the line UV, which passes through the midpoints of BE' and BD' is parallel to $D'E'$.

Editor's Comment. We received a plethora of solutions to this problem with many interesting approaches! In fact, an entire article could probably be written on the number of different viewpoints for this problem.

OC162. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $k, m, n \in \mathbb{N}$ we have

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

Originally problem 6 from the 2012 South Africa National Olympiad.

We received two correct solutions and three incorrect submissions. We present the solution by Oliver Geupel.

The constant function $f : x \mapsto 1$ solves the problem. We prove that there is no other solution.

Let f be any solution of the problem and let $P(k, m, n)$ denote the assertion that $f(km) + f(kn) - f(k)f(mn) \geq 1$. By $P(1, 1, 1)$, we have $(f(1) - 1)^2 \leq 0$, whence $f(1) = 1$. From $P(x, 1, 1)$ we see that $f(x) \geq 1$ for every $x \in \mathbb{N}$. It suffices to show that that there is no x such that $f(x) > 1$.

Assume to the contrary that $f(q) > 1$ for some natural number q. From $P(q, 1, x)$ and $P(1, q, x)$, we obtain

$$f(q) + f(qx) - f(q)f(x) \geq 1, \tag{1}$$
$$f(q) + f(x) - f(qx) \geq 1. \tag{2}$$

Adding the inequalities (1) and (2), we obtain $(f(q) - 1) \cdot (2 - f(x)) \geq 0$, thus,

$$f(x) \leq 2 \quad \text{for every } x \in \mathbb{N}. \tag{3}$$

It is enough to show that for every nonnegative integer n it holds

$$f\left(q^{2^n}\right) - 1 \geq f(q)^n \cdot (f(q) - 1). \tag{4}$$

As a consequence, the values $f\left(q^{2^n}\right)$ increase unboundedly, in contradiction to the bound (3).

We prove (4) by induction on n. The base case $n = 0$ is obvious. Assume (4) holds for some $n = k$. By $P\left(q^{2^k}, q^{2^k}, 1\right)$, we have

$$f\left(q^{2^{k+1}}\right) + f\left(q^{2^k}\right) - f\left(q^{2^k}\right)^2 \geq 1.$$

Notice that $f(q^{2k}) \geq f(q)$ using (4) and $(f(q) - 1)(f(q)^k - 1) \geq 0$. By induction, it follows that

$$f\left(q^{2^{k+1}}\right) - 1 \geq f\left(q^{2^k}\right) \cdot \left(f\left(q^{2^k}\right) - 1\right) \geq f(q) \cdot f(q)^k (f(q) - 1) = f(q)^{k+1}(f(q) - 1),$$

which completes the induction and thus the proof.

Editor's Comment. We received two solutions that assumed that the natural numbers begin at 0 (which is not unreasonable). The problem however becomes substantially easier if we allow for this situation, so the editor chose to display a solution without this concern.

OC163. Let $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ and S be the set of all subsets of A. Find the number of functions $f : S \to B$ such that

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S..$$

Originally problem 1 from day 1 of the 2012 Turkey TST.

We received one solution to this problem. We present Oliver Geupel's solution slightly modified by the editor.

More generally, let $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$. We prove that the number of functions with the required property is $\sum_{k=1}^{n} k^m$.

For $1 \le k \le n$, let F_k denote the set of all functions $f : S \to B$ with the two properties

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S \qquad \text{and} \qquad f(A) = k.$$

Moreover, let G_k denote the set of all functions $g : A \to \{1, 2, \ldots, k\}$.

Note that $|G_k| = k^m$. It is therefore enough to establish bijections

$$F_k \leftrightarrow G_k \qquad (1 \le k \le n).$$

For any $f \in F_k$, let the related $g \in G_k$ be defined by

$$g(j) = f(A \setminus \{j\}) \qquad (1 \le j \le m). \tag{1}$$

Notice that the function is well defined since

$$f(A \setminus \{j\}) = f(A \cap (A \setminus \{j\})) = \min\{f(A), f(A \setminus \{j\}))\} \le k$$

In the following we use the notation $\overline{X} = A \setminus X$. From (1) we deduce by induction on $\left|\overline{X}\right|$ that

$$f(X) = \min\left\{g(j) : j \in \overline{X}\right\} \qquad (X \in S \setminus \{A\}),$$

that is, the m relations (1) determine f uniquely.

Now, starting with a $g \in G_k$, pick a $f \in F_k$ such that (1) holds. Then, for any $A_1, A_2 \in S$, we have

$$\begin{aligned} f(A_1 \cap A_2) &= \min\left\{g(j) : j \in \overline{A_1 \cap A_2}\right\} \\ &= \min\left\{g(j) : j \in \overline{A_1} \cup \overline{A_2}\right\} \\ &= \min\left\{\min\left\{g(j) : j \in \overline{A_1}\right\}, \min\left\{g(j) : j \in \overline{A_2}\right\}\right\} \\ &= \min\{f(A_1), f(A_2)\}, \end{aligned}$$

so that $f \in F_k$.

Hence the map $f \mapsto g$ establishes the desired bijection $F_k \leftrightarrow G_k$.

OC164. Find all triples (m, p, q) where m is a positive integer and p, q are primes such that
$$2^m p^2 + 1 = q^5 \,.$$

Originally problem 3 from day 1 of the 2012 Korean Math Olympiad.

We received five correct solutions and one incorrect submission. We present the solution by Konstantine Zelator.

We will demonstrate that this problem has a unique solution $(m, p, q) = (1, 11, 3)$. Let m, p, q be integers such that $2^m p^2 + 1 = q^5$ where m is a positive integer and p, q are prime numbers. Observe that q must be an odd prime and we can rewrite the equation as

$$2^m p^2 = q^5 - 1 = (q - 1)(q^4 + q^3 + q^2 + q + 1)$$

and note that the factor $q^4 + q^3 + q^2 + q + 1$ is odd. Since

$$q^4 + q^3 + q^2 + q + 1 = (q - 1)(q^3 + 2q^2 + 3q + 4) + 5$$

we see that $\gcd(q-1, q^4+q^3+q^2+q+1) = 1$ or 5. If this greatest common divisor is 5, then the original equation implies that $p = 5$ and hence

$$2^m = \frac{q-1}{5} \cdot \frac{q^4 + q^3 + q^2 + q + 1}{5}.$$

Notice that the factor $\frac{q^4+q^3+q^2+q+1}{5}$ is odd and hence must be 1 since the left hand since has only powers of 2. This would imply that $q = 1$ which is impossible. Thus, the greatest common divisor must be 1.

Now, note that p must also be odd since $q^4 + q^3 + q^2 + q + 1$ is an odd number bigger than 1 and hence has an odd prime divisor. Thus, we must have that

$$q - 1 = 2^m \qquad \text{and} \qquad q^4 + q^3 + q^2 + q + 1 = p^2$$

If $m \geq 3$ then $q \equiv 1 \pmod 8$ and so

$$p^2 = q^4 + q^3 + q^2 + q + 1 \equiv 1^4 + 1^3 + 1^2 + 1 + 1 \equiv 5 \pmod 8$$

which is a contradiction since 5 is not a quadratic residue modulo 8. If $m - 2$, then $q = 2^2 + 1 = 5$ and thus

$$p^2 = q^4 + q^3 + q^2 + q + 1 \equiv 5^4 + 5^3 + 5^2 + 5 + 1 \equiv 1 + 5 + 1 + 5 + 1 \equiv 5 \pmod 8$$

which is also a contradiction. Lastly, if $m = 1$, then $q = 3$ and

$$p^2 = 3^4 + 3^3 + 3^2 + 3 + 1 = 121$$

and thus $p = 11$. Hence, we obtain the unique solution to this problem $(m, p, q) = (1, 11, 3)$.

OC165. Let O be the circumcenter of acute ΔABC, and let H be its orthocenter. Let $AD \perp BC$, and let EF be the perpendicular bisector of AO, with D, E on the side BC. Prove that the circumcircle of ΔADE passes through the midpoint of OH.

Originally problem 1 from day 2 of the 2012 China Western Olympiad.

We received six correct solutions. We present the solution by Somasundaram Muralidharan.

Remark. The following solution uses complex slope and complex equations of lines. The basic facts are as follows:

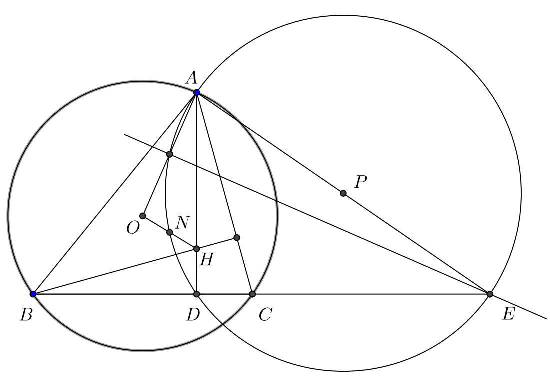
For points $A(z_1)$ and $B(z_2)$, the slope of the line AB is given by $\kappa = \frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}}$. If the line makes an angle θ with the positive real axis, then the slope of the line is the complex number $e^{2i\theta}$.

The equation of the line AB is given by $z - z_1 = \kappa(\bar{z} - \bar{z_1})$. Two lines with slopes κ_1 and κ_2 are parallel if and only if $\kappa_1 = \kappa_2$ and they are perpendicular if and only if $\kappa_1 + \kappa_2 = 0$.

If $A(z_1)$ and $B(z_2)$ lie on the unit circle, $|z| = 1$, then the slope of the line AB is given by

$$\frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}} = \frac{z_1 - z_2}{\frac{1}{z_1} - \frac{1}{z_2}} = -z_1 z_2.$$

For more details, see *P.S.Modenov, Problems in Geometry, MIR Publishers, 1981.*



There is no loss of generality in assuming that the circumcenter O is at the origin and that the circumradius is 1. Let the complex numbers representing the vertices A, B, C be respectively a, b, c. Then the orthocenter H is represented by the complex number $a + b + c$ and N, the mid point of OH is the complex number $\frac{a+b+c}{2}$.

For any point R, let $\bar{R}$ represent the reflection of the point R on the real axis. If z is the complex number representing R, then the complex number representing

$\bar{R}$ will be the complex conjugate of z, represented by $\bar{z}$.

The slope of the line OA is $\dfrac{a}{\bar{a}} = a^2$. The equation of the perpendicular bisector of OA is

$$Z - \frac{a}{2} = -a^2\left(\bar{Z} - \frac{\bar{a}}{2}\right). \tag{1}$$

Simplifying (1), we obtain the equation of the perpendicular bisector of OA as

$$Z - a = -a^2\bar{Z}. \tag{2}$$

The equation of the line BC is

$$Z - b = -bc(\bar{Z} - \bar{b})$$

which can be written as

$$Z - b - c = -bc\bar{Z}. \tag{3}$$

We can assume that OA is not perpendicular to BC, since, in this case, the above two lines are parallel. Since the slopes are $-a^2$ and $-bc$, the condition that these two intersect is $-a^2 \neq -bc$ or $a^2 - bc \neq 0$.

Solving (2) and (3) we can obtain the point E. From (2) and (3), we get

$$\bar{Z} = \frac{a - b - c}{a^2 - bc}.$$

This represents the point $\bar{E}$, the reflection of E on the real axis. If P is the center of the circle ADE, then P is the mid point of AE (since $\angle ADE = 90°$). The point $\bar{A}$ is given by $\bar{a}$ and hence $\bar{P}$ is given by

$$\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} + \bar{a}\right).$$

Thus the radius of the circle ADE is

$$\begin{aligned} \rho = |AP| = |\bar{A}\bar{P}| &= \left|\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} + \bar{a}\right) - \bar{a}\right| \\ &= \left|\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} - \bar{a}\right)\right| \\ &= \frac{1}{2}\left|\frac{\bar{a}bc - b - c}{a^2 - bc}\right|. \end{aligned}$$

Now, it suffices to show that the distance between $\bar{N}$ and $\bar{P}$ is also equal to ρ.

Then,

$$\begin{aligned}
|\bar{N}\bar{P}| &= \left|\frac{1}{2}\left(\frac{a-b-c}{a^2-bc}+\bar{a}\right)-\left(\frac{\bar{a}+\bar{b}+\bar{c}}{2}\right)\right| \\
&= \frac{1}{2}\left|\frac{a-b-c}{a^2-bc}-\bar{b}-\bar{c}\right| \\
&= \frac{1}{2}\left|\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \\
&= \frac{1}{2}|\bar{a}^2bc|\left|\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \qquad \text{since } |a|=|b|=|c|=1 \\
&= \frac{1}{2}\left|(\bar{a}^2bc)\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \\
&= \frac{1}{2}\left|\frac{\bar{a}bc-c-b}{a^2-bc}\right|.
\end{aligned}$$

Hence $|\bar{N}\bar{P}| = \rho$ and the circle ADE passes through N, the midpoint of OH.