Application of Inversive Methods to Euclidean Geometry

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Place a sphere on top of the Euclidean plane so that its south pole S is at the origin. Let N be the north pole. For any point $Q \neq N$ on the sphere, the point P of intersection of the extension of NQ with the plane is called its image under the **stereographic projection** from N.

Of course, it would be tidier if N had an image as well. How would it behave? As Q approaches N from any direction, the projection P "approaches infinity" in the sense of becoming arbitrarily far away from S. If we add a *point at infinity* I to the Euclidean plane, with the property that a sequence (P_j) is defined to converge to I if and only if $|P_j|$ increases without bound, we will have what is known as the **inversive plane**. Think of the sphere as a balloon and the point N as a puncture. If we stretch the balloon out onto the plane, we can see that the point I is in every direction!

We define the point I to be the projection of N. It is called the **ideal point**, and lies on every straight line. To see this, consider a straight line ℓ on the inversive plane and the plane passing through N and ℓ. The cross-section with the sphere is a circle passing through the point N, justifying the statement that I lies on every straight line. In fact, it closes the straight line into something like a circle.

Inversion

For any circle Σ with center O and radius R, and any point $A \neq O, I$, we define the *inverse point of A with respect to* Σ to be the point on the ray $\overrightarrow{OA}$ at distance $R^2/|OA|$ from O. This is readily seen to be an involution (self-inverse map). The points O and I are defined to invert into each other. We consider straight lines to be "circles passing through I". Inversion in a straight line is defined to be reflection: the point I is fixed under reflections. The geometry resulting from (and preserved by) these mappings is called *inversive geometry.* For a full introduction to inversive geometry, the reader is referred to any good undergraduate geometry textbook, such as Pedoe [1] (chapter VI) or Baragar [2] (chapter 7).

Exercise 1 *Inversion fixes exactly the points of Σ. It maps points inside Σ to points outside Σ and* vice versa.

The next result is a very useful lemma. Note the order in which the points of the triangles are specified - this is important!

Exercise 2 *Let P, Q, and the center of inversion O not be collinear, and let P,Q invert to P',Q'. Then the triangles $\triangle OPQ$ and $\triangle OQ'P'$ are similar.*

The reflection of a circle in a line is always a circle. Something similar is true for inversions.

Exercise 3 *Inversion maps circles not passing through O to circles, circles passing through O to straight lines not through O, straight lines passing through O to themselves, and other straight lines to circles.*

We can define the angle between two circles, or between a circle and a line, at a point P to be the angle between the tangent lines. Reflection preserves these, of course — so does inversion.

Exercise 4 *Show that inversion preserves angles, whether between two lines, a line and a circle, or two circles.*

Reflection in a line L maps any line or circle that is orthogonal to L to itself. (Note that if a circle meets a line or another circle twice, it makes the same angle at each intersection point. Thus "orthogonal" is well defined here and in the following exercise.)

Exercise 5 *Show that inversion maps any circle orthogonal to Σ to itself.*

Exercise 6 *If a circle C cuts Σ, so does its inverse. If a circle C is tangent to Σ, so is its inverse. If a circle C contains O in its interior, so does its inverse.*

Exercise 7 *The Euclidean construction for an inverse point is simple enough to find by trial and error.*

- *(i) Given O and Σ, and a point P inside Σ, construct the inverse point P'.*
- *(ii) Given O and Σ, and a point P outside Σ, construct the inverse point P'.*

Reflection preserves reflections: that is, a mirror seen in a mirror acts like a mirror. Something similar holds for inversions:

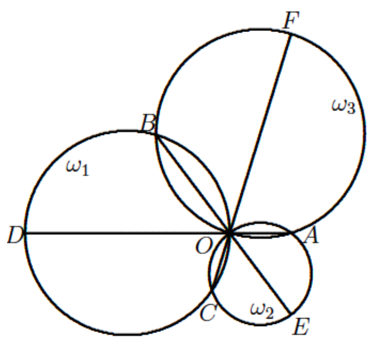
Exercise 8 *If P and P' are inverse with respect to C, and their inverses with respect to Σ are $\overline{P}$,$\overline{P'}$, and $\overline{C}$ respectively, then $\overline{P}$ and $\overline{P'}$ are inverses with respect to $\overline{C}$.*

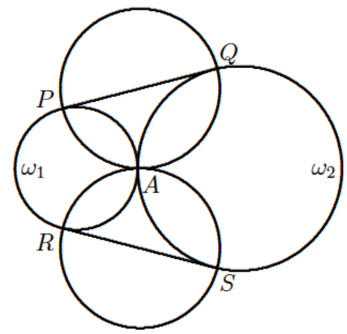
We might wonder if inversion preserves circle centers, but it doesn't. (It's easy to find a counterexample — find one!) There is a way to find the center of an inverse circle, though.

Exercise 9 *If C and $\overline{C}$ are inverses with respect to Σ, then the center A of $\overline{C}$ is found as follows. Let B be the inverse of O in C; then A is the inverse of B in Σ.*

Problems

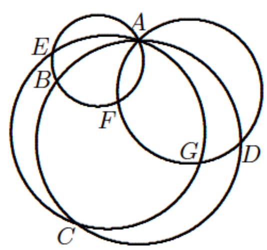
Problem 1 (below left)
Three circles ω_1, ω_2 and ω_3 pass through O. C is the other point of intersection of ω_1 and ω_2, A is the other point of intersection of ω_2 and ω_3, and B is the other point of intersection of ω_3 and ω_1. The extension of AO intersects ω_1 again at D, the extension of BO intersects ω_2 again at E, and the extension of CO intersects ω_3 again at F. Prove that if OE and OF are diameters of ω_2 and ω_3 respectively, then OD is a diameter of ω_1.

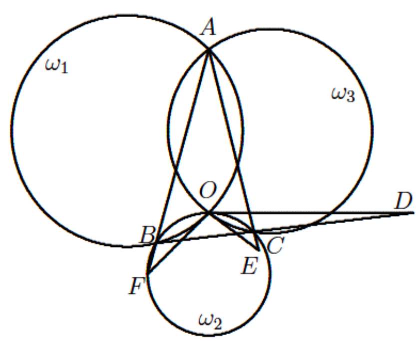




Problem 2 (above right)
Two circles ω_1 and ω_2 are tangent externally to each other at A. A common exterior tangent touches ω_1 at P and ω_2 at Q. The other common exterior tangent touches ω_1 at R and ω_2 at S. Prove that the circumcircles of triangles PAQ and RAS are tangent to each other.

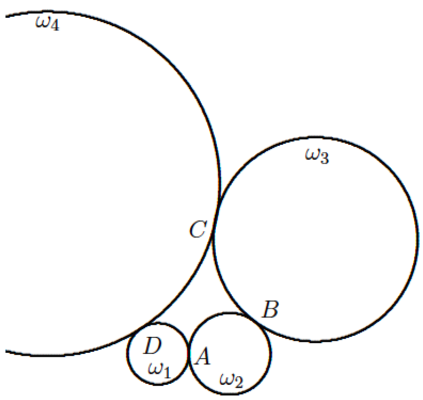
Problem 3 (below left)
AB, AC and AD are three chords on a circle. Circles with AB and AC as diameters intersect at E, circles with AB and AD as diameters intersect at F, and circles with diameters AC and AD intersect at G. Prove that E, F and G are collinear.

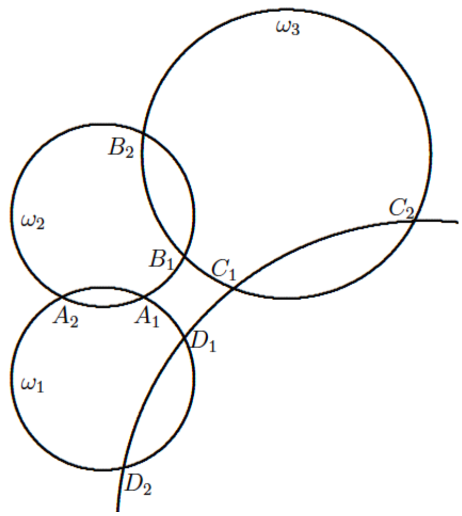




Problem 4 (above right)
Three circles ω_1, ω_2 and ω_3 pass through O. B is the other point of intersection of ω_1 and ω_2, C is the other point of intersection of ω_2 and ω_3, and A is the other point of intersection of ω_3 and ω_1. The tangent to ω_2 at O intersects BC at D, the tangent at O to ω_3 intersects CA at E, and the tangent at O to ω_1 intersects AB at F. Prove that D, E and F are collinear.

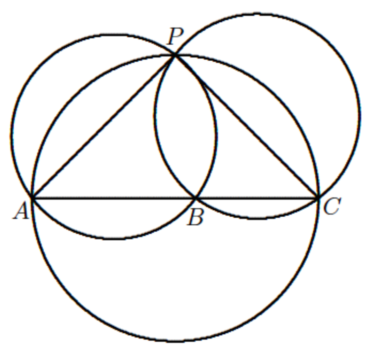
Problem 5 (below left)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 touch at A, ω_2 and ω_3 touch at B, ω_3 and ω_4 touch at C and ω_4 and ω_1 touch at D. Prove that A, B, C and D are concyclic.





Problem 6 (above right)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 intersect at A_1 and A_2, ω_2 and ω_3 intersect at B_1 and B_2, ω_3 and ω_4 intersect at C_1 and C_2, and ω_4 and ω_1 intersect at D_1 and D_2. Prove that if A_1, B_1, C_1 and D_1 are collinear or concyclic, then so are A_2, B_2, C_2 and D_2.

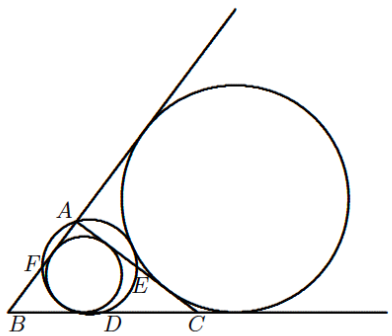
Problem 7 (below)
A, B and C are three points on a line and P is a point not on this line. Prove that the circumcentres of triangles PAB, PBC and PCA are concyclic with P.

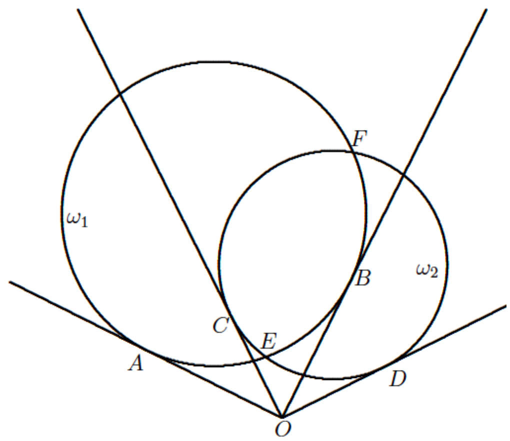


Problem 8
Prove Ptolemy's Inequality which states that $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ for any convex quadrilateral $ABCD$, with equality if and only if the quadrilateral is cyclic. (Hint: Because this is quantitative, expect to use the "polar-coordinate" definition of inversion.)

Problem 9 (below left)
Prove that the circle which passes through the midpoints of the sides of a triangle

is tangent to the triangle's incircle and excircles.

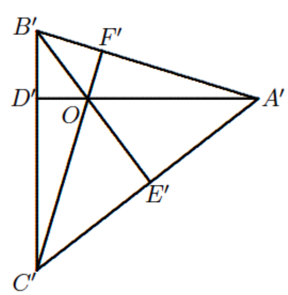




Problem 10 (above right)
From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.

The solution to Problem 1 is given as an example. We leave the others to the reader!

Solution (to Problem 1) Invert with respect to any circle with center O. Then the three circles turn into triangle $A'B'C'$ while the radial lines OA, OB and OC invert to themselves. That OE is a diameter of ω_2 means that $B'E'$ is orthogonal to $A'C'$. Similarly, $C'F'$ is orthogonal to $A'B'$. Hence O is the orthocentre of triangle $A'B'C'$, so that $A'O$ is orthogonal to $B'C'$. It follows that OD is indeed a diameter of ω_1.



References

[1] Baragar, A., *A Survey of Classical and Modern Geometries*, Pearson, 2001.

[2] Pedoe, D., *Geometry: A Comprehensive Course*, Dover, 1970.