FOCUS ON...

No. 16

Michel Bataille

Leibniz's and Stewart's relations

Introduction

Let $A_1, A_2, \ldots, A_n$ be points in space and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. A quick study of the sum

$$\ell(M) = \alpha_1 MA_1^2 + \alpha_2 MA_2^2 + \cdots + \alpha_n MA_n^2$$

where M is an arbitrary point, will lead to Leibniz's relation, a result often proved as a lemma when needed and which certainly deserves to be better known. As a by-product, we will obtain Stewart's relation, naturally linked to this study.

Transformation of $\ell(M)$

We introduce the vectorial version of ℓ defined by

$$\overrightarrow{\mathcal{L}}(M) = \alpha_1 \overrightarrow{MA_1} + \alpha_2 \overrightarrow{MA_2} + \cdots + \alpha_n \overrightarrow{MA_n}$$

which obviously satisfies $\overrightarrow{\mathcal{L}}(M) - \overrightarrow{\mathcal{L}}(N) = \alpha\overrightarrow{MN}$ where $\alpha = \sum_{k=1}^{n} \alpha_k$ for all M, N. From this relation, it follows that $\overrightarrow{\mathcal{L}}(M)$ is a vector $\overrightarrow{U}$ independent of M when $\alpha = 0$ and otherwise that $\overrightarrow{\mathcal{L}}(G) = \overrightarrow{0}$ for a unique point G, the centre of mass of the weighted points (A_k, α_k), $k = 1, 2, \ldots, n$. Note that in the latter case, $\overrightarrow{\mathcal{L}}(M) = \alpha\overrightarrow{MG}$ for any point M.

Now, using $MA^2 - NA^2 = (\overrightarrow{MA} - \overrightarrow{NA}) \cdot (\overrightarrow{MA} + \overrightarrow{NA})$, we readily obtain

$$\ell(M) - \ell(N) = \overrightarrow{MN} \cdot (\overrightarrow{\mathcal{L}}(M) + \overrightarrow{\mathcal{L}}(N)).$$

This gives $\ell(M) - \ell(N) = 2\overrightarrow{MN} \cdot \overrightarrow{U}$ when $\alpha = 0$ and, if $\alpha \neq 0$, what is generally called Leibniz's relation: $\ell(M) = \ell(G) + \overrightarrow{MG} \cdot \overrightarrow{\mathcal{L}}(M) = \ell(G) + \alpha MG^2$, that is,

$$\alpha_1 MA_1^2 + \alpha_2 MA_2^2 + \cdots + \alpha_n MA_n^2 = \alpha MG^2 + \alpha_1 GA_1^2 + \alpha_2 GA_2^2 + \cdots + \alpha_n GA_n^2.$$

To see these formulas at work in elementary examples, consider two distinct points B, C. Denoting by I the midpoint of BC and by A an arbitrary point, Leibniz's relation gives $AB^2 + AC^2 = 2AI^2 + IA^2 + IB^2$ leading to $4AI^2 = 2AB^2 + 2AC^2 - BC^2$, the familiar formula for the median AI in triangle ABC. On the other hand, consider the equality $MB^2 - MC^2 = AB^2 - AC^2$ for some point M. This relation rewrites as $(MB^2 - MC^2) - (AB^2 - AC^2) = 0$ or $2\overrightarrow{MA} \cdot \overrightarrow{CB} = 0$ (here $\alpha = 0$ and

$\overrightarrow{U} = \overrightarrow{CB}$). Thus, the locus of the points M such that $MB^2 - MC^2 = AB^2 - AC^2$ is the plane through A orthogonal to BC.

Two applications

As a first application, we consider problem 11433 of *The American Mathematical Monthly* (slightly reformulated):

> Let n be a positive integer, and $A_1, \dots, A_n, B_1, \dots, B_n$, and $C_1, \dots, C_n$ be points on the unit sphere S^2. Show that there exists P on S^2 such that
> $$\sum_{k=1}^{n} PA_k^2 = \sum_{k=1}^{n} PB_k^2 = \sum_{k=1}^{n} PC_k^2.$$

Call E the centre of mass of $(A_1, 1), \dots, (A_n, 1)$. From Leibniz's relation, we have

$$\sum_{k=1}^{n} MA_k^2 = nME^2 + \sum_{k=1}^{n} EA_k^2$$

for any point M in space. Taking M at O, the centre of S^2, we obtain

$$\sum_{k=1}^{n} EA_k^2 = n(1 - OE^2)$$

and so

$$\sum_{k=1}^{n} MA_k^2 = n(1 + ME^2 - OE^2).$$

Similarly, if F is the centre of mass of $(B_1, 1), \dots, (B_n, 1)$ and G the one of $(C_1, 1), \dots, (C_n, 1)$, the following equalities hold:

$$\sum_{k=1}^{n} MB_k^2 = n(1 + MF^2 - OF^2), \qquad \sum_{k=1}^{n} MC_k^2 = n(1 + MG^2 - OG^2).$$

Now, let $\mathcal{P}_1 = \{M : ME^2 - MF^2 = OE^2 - OF^2\}$ and $\mathcal{P}_2 = \{M : MF^2 - MG^2 = OF^2 - OG^2\}$. We observe that a suitable point P is a point of S^2 which also belongs to $\mathcal{P}_1 \cap \mathcal{P}_2$. The existence of such a point P is ensured by the following discussion.

• If $E = F = G$, every point of S^2 is suitable.

• If, say, $E \neq F$ and $F = G$, then $\mathcal{P}_1 = \mathcal{P}_1 \cap \mathcal{P}_2$ is the plane through O orthogonal to the line EF. Any point of the great circle intersection of this plane with S^2 is suitable. The same conclusion holds if E, F, G are distinct and collinear since then $\mathcal{P}_1 = \mathcal{P}_2$.

• In the general case where E, F, G are not collinear, $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct planes which both pass through O. They intersect along a diameter of S^2 and this diameter intersects S^2 in two suitable points.

Our second example is provided by Christopher Bradley's problem **2629** [2001 : 214 ; 2002 : 256]:

> In triangle ABC, the symmedian point is denoted by S. Prove that
> $$\frac{1}{3}(AS^2+BS^2+CS^2) \geq \frac{BC^2AS^2+CA^2BS^2+AB^2CS^2}{BC^2+CA^2+AB^2}.$$

We propose the following variant of Joel Schlosberg's featured solution.

Let $BC=a, CA=b, AB=c$ and let G be the centroid of ΔABC. Since S is the centre of mass of $(A,a^2),(B,b^2),(C,c^2)$ and G is the centre of mass of $(A,1),(B,1),(C,1)$, Leibniz's relation yields on the one hand

$$a^2GA^2+b^2GB^2+c^2GC^2=(a^2+b^2+c^2)GS^2+a^2SA^2+b^2SB^2+c^2SC^2$$

and on the other hand,

$$AS^2+BS^2+CS^2=3GS^2+AG^2+BG^2+CG^2.$$

It easily follows that the required inequality is equivalent to

$$6(a^2+b^2+c^2)GS^2 \geq (2a^2-b^2-c^2)GA^2+(2b^2-c^2-a^2)GB^2+(2c^2-a^2-b^2)GC^2 \quad (1)$$

Using $GA^2=\frac{1}{9}(2b^2+2c^2-a^2)$ *etc.*, the right-hand side R of (1) becomes after some calculations, $R=-\frac{1}{3}((a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2)$, hence $R \leq 0$ and (1) follows.

Note that equality holds if and only if the triangle ABC is equilateral.

Stewart's relation

Equality (2) below, generally called Stewart's relation, relates four points, three of which are collinear.

> Let A,B,C be points on a line δ and let M be an arbitrary point. Then
> $$\overline{BC}.MA^2+\overline{CA}.MB^2+\overline{AB}.MC^2+\overline{BC}.\overline{CA}.\overline{AB}=0 \qquad (2)$$
> where $\overline{XY}$ denotes the signed distance from X to Y.

The proof is a nice application of the case $\alpha=0$ in the general study of $\ell(M)$.

Let $\ell(M)=\overline{BC}.MA^2+\overline{CA}.MB^2+\overline{AB}.MC^2$. Since $\overline{BC}+\overline{CA}+\overline{AB}=0$, we have $\ell(M)-\ell(N)=2\overrightarrow{MN}\cdot\overrightarrow{U}$ where $\overrightarrow{U}$ is the corresponding constant vector

$$\overrightarrow{U}=\overrightarrow{\mathcal{L}}(M)=\overline{BC}\,\overrightarrow{MA}+\overline{CA}\,\overrightarrow{MB}+\overline{AB}\,\overrightarrow{MC}=\overline{CA}\,\overrightarrow{AB}+\overline{AB}\,\overrightarrow{AC}=\overrightarrow{0}.$$

Thus, $\ell(M)$ is independent of M as well, and so $\ell(M)=\ell(A)=\overline{CA}.\overline{AB}.\overline{CB}$, as desired.

The application coming to mind at once is to the cevians of a triangle:

Let $D = tB + (1-t)C$ be a point of the sideline BC of triangle ABC ($t \in \mathbb{R}$). Then, the length of the cevian AD is given by

$$AD^2 = tc^2 + (1-t)b^2 - t(1-t)a^2 \qquad (3)$$

where, as usual, $a = BC, b = CA, c = AB$.

This directly follows from (2) with $M = A$ and points B, D, C on line BC. Indeed, we first get

$$\overline{DC}.AB^2 + \overline{BD}.AC^2 + \overline{CB}.AD^2 + \overline{DC}.\overline{BD}.\overline{CB} = 0$$

and then, since $\overline{BD} = (1-t)\overline{BC},\ \overline{DC} = t\overline{BC}$,

$$t\overline{BC}.c^2 + (1-t)\overline{BC}.b^2 - \overline{BC}.AD^2 - t\overline{BC}.(1-t)\overline{BC}.\overline{BC} = 0.$$

Dividing by $\overline{BC}$ yields (3).

In particular, if AD is the internal bisector of $\angle BAC$, then $t = \dfrac{b}{b+c}$ and so

$$AD^2 = \frac{b}{b+c}\cdot c^2 + \frac{c}{b+c}\cdot b^2 - a^2 \cdot \frac{bc}{(b+c)^2}.$$

This leads to the known formula $AD^2 = bc - \dfrac{a^2bc}{(b+c)^2}$.

For another particular case, consider the symmedian point S that we met earlier. If AS intersects BC at D, we obtain the length of the symmedian from A:

$$AD = \frac{2bcm_a}{b^2+c^2}$$

where m_a is the length of the median from A; this follows from (3) with $t = \dfrac{b^2}{b^2+c^2}$.

We conclude with two exercises.

Exercises

1. Let P be an arbitrary point in the plane of a triangle ABC with sidelengths a, b, c. Prove that

$$PA^2 + PB^2 + PC^2 \geq \frac{a^2+b^2+c^2}{3}.$$

2. Let A, B, C, D be four points on a line ℓ in this order and let M not on ℓ be such that $\angle AMB = \angle CMD$. Prove that

$$\frac{MA^2}{MC^2} > \frac{AB}{CD} > \frac{MB^2}{MD^2}.$$