Crux Mathematicorum

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Editorial Board

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EDITORIAL

Have you ever heard of the airplane refuelling problem? I've just recently stumbled upon it in a research paper. Just like many problems that grew into a research topic, this problem started off as a puzzle originally appearing in *Puzzle Math* by George Gamow and Marvin Stern. The simplest version of the problem is as follows: suppose you have to fly a plane non-stop around the world, the distance that is greater than the range of any plane you have available. In fact, you can only go half way around the world on a full tank, so you will have to arrange other planes to assist you with refuelling mid-air. Suppose you start with several identical planes that can all refuel each other, how many planes do you need and what is your refuelling plan? Remember: all the planes have to be able to return home!

It is easy to make this problem more complicated, which is of course exactly what mathematicians did. What if the full tank only takes you $1/n^{th}$ of the way? What if the planes are not identical? How about if you want to account for refuelling time and decreasing fuel consumption? This most general airplane refuelling problem with arbitrary tank volumes and consumption rates now fits into the class of problems known as scheduling problems and is still open although some yet unpublished results claim differently.

This is the beauty of problem solving: it starts off as a game, a puzzle, where the participant is guided only by his or her own curiosity, and ends . . . well, it need not end. This is what ***Crux*** is for. I'll leave you with this quote by John von Neumann, whose birthday it happens to be today as I'm writing this (he would have been 112 years old):

> A large part of mathematics which becomes useful developed with absolutely no desire to be useful, and in a situation where nobody could possibly know in what area it would become useful; and there were no general indications that it ever would be so. By and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and the moment when it is useful; and that this lapse of time can be anything from 30 to 100 years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do things which are useful. (In "The Role of Mathematics in the Sciences and in Society", an address to Princeton alumni in 1954.)

Kseniya Garaschuk

THE CONTEST CORNER

No. 33

Robert Bilinski

The problems featured in this section have appeared in, or have been inspired by, a mathematics contest question at either the high school or the undergraduate level. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

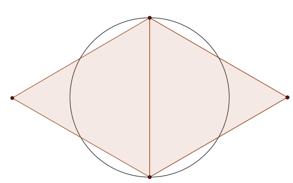
To facilitate their consideration, solutions should be received by the editor by **May 1, 2016**, *although late solutions will also be considered until a solution is published.*

CC161. A number n written in base b reads 211, but it becomes 110 when written in base $b+2$. Find n and b in base 10.

CC162. What is the probability that 99 divides a randomly chosen 4-digit palindrome?

CC163. If x is randomly chosen in $[-100, 100]$, what is the probability that $g[f(x)]$ is negative given that $f(x) = x^2 + 3x - 7$ and $g(x) = x^2 - 2x - 99$?

CC164. Build two equilateral triangles on the diameter of a circle with radius 5. What is the total area of the circle outside the equilateral triangles?



CC165. Georges pays \$50 on each of four gas refills but the prices per litre were \$1.32, \$1.25, \$1.11 and \$1.18 as the price was fluctuating a lot in that time period. What is the average price per litre?

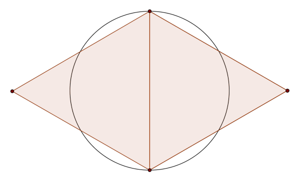
. . . .

CC161. Un nombre n écrit en base b se lit 211, mais devient 110 quand on l'écrit dans la base $b+2$. Trouver n et b en base 10.

CC162. Quelle est la probabilité que 99 divise un palindrome à 4 chiffres choisi au hasard?

CC163. Si on choisit x aléatoirement dans $[-100, 100]$, quelle est la probabilité que $g[f(x)]$ est négatif compte tenu que $f(x) = x^2 + 3x - 7$ et $g(x) = x^2 - 2x - 99$?

CC164. Sur le diamètre d'un cercle de rayon 5, on construit 2 triangles équilatéraux. Quelle est l'aire totale du cercle en dehors des triangles équilatéraux?



CC165. Georges paye 50\$ à chacune de quatre visites à une station-service lorsque les prix par litre étaient 1,32 \$, 1,25 \$, 1,11 \$ et 1,18 \$ pendant une période de grande fluctuation de prix. Quel était le prix moyen par litre?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(3), p. 96–97.

CC111. Find all positive integers with two or more digits such that if we insert a 0 between the units and tens digits we get a multiple of the original number.

Originally problem A1 from the 2003 Mexican Math Olympiad.

We received ten correct submissions. We present the solution by Yihang Dong.

Call the number $\overline{xy}$, where y is the one's digit and x is the rest. Then we are looking for numbers such that $k(10x+y) = 100x+y$. If $k > 10$, then $10kx > 100x$ and $ky > y$, which cannot happen. So $k \leq 10$.

Considering mod 10, we see that $ky \equiv y \pmod{10}$. So $10a = ky - y$ for some a. One possibility is $y = 0$. If $y \neq 0$, then $(k-1)y = 10a$. So $10 \mid (k-1)y$. If $5 \mid y$, then $y = 5$. If $5 \mid (k-1)$, then $k - 1 = 5$ and so $k = 6$.

Let us go through these three possibilities.

1. If $y = 0$, then $10kx = 100x$ has solution $k = 10$ for all integers x. So any multiple of 10 works.

2. If $y = 5$, then $k(10x + 5) = 100x + 5$. And so $x = \frac{k-1}{20-2k}$, which means that $k - 1 \geq 20 - 2k$, giving $k \geq 7$. We also have that $2 \mid (k - 1)$, so $k = 7$ or $k = 9$. These lead to $x = 1$ or $x = 4$, leading to solutions 15 and 45.

3. If $k = 6$, then $60x + 6y = 100x + y$, or rather $y = 8x$. Thus $x = 1, y = 8$ and we get the solution 18.

In conclusion, the possibilities are $15, 18, 45$ and multiples of 10.

CC112. Jerome groups odd numbers in groups that contain successive quantities of odd number of elements such as:

$$\{1\}, \{3, 5\}, \{7, 9, 11\}, \{13, 15, 17, 19\}, \ldots$$

What is the sum of the 100th grouping?

Inspired by problem 13 from the second round of South African school Olympiads 2014.

We received twelve correct solutions and two incorrect submissions. We present the solution by Matei Coiculescu.

Because the number of elements in each group increases by 1, for every group, the number of elements until the 100th grouping is

$$1 + 2 + 3 + \cdots + 99 = \frac{99 \cdot 100}{2} = 4950.$$

This means that the sum of the elements of the 100th grouping is the sum of the odd numbers from the 4951st odd number to the 5050th odd number. This sum can be expressed as

$$\sum_{k=4951}^{5050} (2k - 1) = \sum_{k=1}^{5050} (2k - 1) - \sum_{k=1}^{4950} (2k - 1) = 5050^2 - 4950^2 = 1000000.$$

CC113. If P is a point inside $ABCD$ with $PA = 2$, $PB = 3$, $PC = 5$ and $PD = 6$, what is the maximum possible area of $ABCD$?

Originally problem 20 from the qualifying round of South African school Olympiads 2012.

We received five correct solutions, and one incomplete submission. We present two solutions: one starting from the sine formula for area (a feature of all but one of the correct solutions), and one using cross products.

Solution 1, by Hessami Pilehroon Elnaz.

The area of quadrilateral $ABCD$ is equal to

$$\begin{aligned}[APB] + [BPC] + [CPD] + [DPA] \\ &= \frac{AP \cdot PB \cdot \sin\angle APB}{2} + \frac{BP \cdot PC \cdot \sin\angle BPC}{2} \\ &+ \frac{CP \cdot PD \cdot \sin\angle CPD}{2} + \frac{DP \cdot AP \cdot \sin\angle DPA}{2}\end{aligned}$$

Substituting $AP = 2$, $BP = 3$, $CP = 5$ and $DP = 6$, we get

$$[ABCD] = 3\sin\angle APB + 7.5\sin\angle BPC + 15\sin\angle DPC + 6\sin\angle APD.$$

The maximum value of $\sin\alpha$ is 1 and occurs when $\alpha = 90^\circ$. Therefore, the maximum value of $[ABCD]$ is $3+7.5+15+6 = 31.5$, and it occurs when $\angle APB = \angle BPC = \angle DPC = \angle APD = 90^\circ$. The four 90° angles make 360° at P, so such a quadrilateral can be constructed as follows: draw two perpendicular lines intersecting at P, and measure the correct distances from P to determine the vertices A, B, C and D. Hence the maximum area of $ABCD$ is 31.5.

Solution 2, by Somasundaram Muralidhanan.

Let P be the origin of vectors and let $\overrightarrow{PA} = \mathbf{a}$, $\overrightarrow{PB} = \mathbf{b}$, $\overrightarrow{PC} = \mathbf{c}$ and $\overrightarrow{PD} = \mathbf{d}$. The area of triangle APB, for example, is then $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$, where $\times$ denotes the cross product and $\|\cdot\|$ the vector norm. Adding up the area of the triangles which make up $ABCD$, we get

$$S = \frac{1}{2}\left(\|\mathbf{a}\times\mathbf{b}\| + \|\mathbf{b}\times\mathbf{c}\| + \|\mathbf{c}\times\mathbf{d}\| + \|\mathbf{d}\times\mathbf{a}\|\right).$$

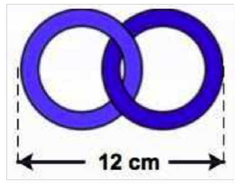
Note that all the cross product vectors in the above expression point in the same direction, so we can combine the norms; that is,

$$\begin{aligned}S &= \frac{1}{2}\|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{d} + \mathbf{d}\times\mathbf{a}\| \\ &= \frac{1}{2}\|\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{d} + \mathbf{c}\times\mathbf{d} - \mathbf{c}\times\mathbf{b}\| \\ &= \frac{1}{2}\|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{d})\| \\ &= \frac{1}{2}\|\mathbf{a}-\mathbf{c}\|\cdot\|\mathbf{b}-\mathbf{d}\|\cdot\sin\theta\end{aligned}$$

where θ is the angle between AC and BD. Clearly, $\|\mathbf{a}-\mathbf{c}\|$ is maximum when P, A, C are collinear and $\|\mathbf{b}-\mathbf{d}\|$ is maximum when P, B, D are collinear. In addition, since the maximum value for $\sin\theta$ is 1, it follows that S is maximum when AC is perpendicular to BD. Thus the maximum area is

$$\frac{1}{2}(2\cdot 3 + 3\cdot 5 + 5\cdot 6 + 6\cdot 2) = \frac{63}{2}.$$

CC114. A chain with two links is 12 cm long. A chain with five links is 27 cm long. What is the length, in cm, of a chain with 40 links?



Originally problem 19 from 2012 South African school Olympiads.

We received nine correct solutions. We present the solution by Yihang Dong.

Let a be the diameter of a link, and let b be the overlap of 2 links. Then we have the system of equations

$$\begin{aligned} 2a - b &= 12 \\ 5a - 4b &= 27, \end{aligned}$$

which has solution $a = 7$ and $b = 2$. A chain with 40 links will have 39 overlaps, leading to $40a - 39b = 202$ cm. Therefore a 40 link chain will be 202 cm long.

CC115. Mathias has put together 120 identical unit cubes to form a rectangular prism and painted all six sides of it. There are 24 unpainted cubes left when the prism is undone. What is the surface area of the prism?

Originally problem 20 from 2015 South African school Olympiads.

We received four correct solutions and one incorrect submission. We present the solution by David Lowry-Duda.

We think of the prism as an inner core of 24 cubes in a prism surrounded by a shell of the remaining cubes in a larger prism.

Then if the dimensions of the inner core are $lwh = 24$, the dimensions of the entire prism are $(l+2)(w+2)(h+2) = 120$. Note that $120 = 2^3 \cdot 3 \cdot 5$ and $24 = 2^3 \cdot 3$. Then we must choose l, w, h from the factors of 24 so that both $lwh = 24$ and $(l+2)(w+2)(h+2) = 120$ with extended side lengths appearing as factors of 120.

Since 5 must divide one of $l+2, w+2, h+2$, we can conclude without loss of generality that $l = 3$, as this is the only way to yield exactly one multiple of 5 (the other case produces no integer solutions). Then $wh = 8$ and $(w+2)(h+2) = wh + 2w + 2h + 4 = 24$, or rather $2w + 2h = 12$. Since the possibilities for h, w are either $1, 2^3$ or $2, 2^2$ and $2w + 2h = 12$, we must have (without loss of generality) that $w = 4, h = 2$.

So the inner core is a $2 \times 3 \times 4$ prism, and the full prism is $4 \times 5 \times 6$. The surface area is $2 \cdot (20) + 2 \cdot (30) + 2 \cdot (24) = 148$.

THE OLYMPIAD CORNER

No. 331

Carmen Bruni

The problems featured in this section have appeared in a regional or national mathematical Olympiad. Readers are invited to submit solutions, comments and generalizations to any problem. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **May 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

OC221. From the point P outside a circle ω with center O draw the tangents PA and PB where A and B lie on ω. From a random point M on the chord AB, we draw the perpendicular to OM, which intersects PA and PB in C and D, respectively. Prove that M is the midpoint of CD.

OC222. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a + b$. Prove that the quotient is at most $\frac{a+b}{4}$. When is this quotient exactly equal to $\frac{a+b}{4}$?

OC223. Let $\mathbb{Z}$ be the set of integers. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

for all $x, y \in \mathbb{Z}$ with $x \neq 0$.

OC224. Let $n > 1$ be an integer. An $n \times n$-square is divided into n^2 unit squares. Of these unit squares, n are coloured green and n are coloured blue, and all remaining ones are coloured white. Are there more such colourings for which there is exactly one green square in each row and exactly one blue square in each column; or colourings for which there is exactly one green square and exactly one blue square in each row?

OC225. Find the maximum value of real number k such that

$$\frac{a}{1 + 9bc + k(b - c)^2} + \frac{b}{1 + 9ca + k(c - a)^2} + \frac{c}{1 + 9ab + k(a - b)^2} \geq \frac{1}{2}$$

holds for all non-negative real numbers $a,\ b,\ c$ satisfying $a + b + c = 1$.

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OC221. À partir d'un point P à l'extérieur d'un cercle ω de centre O, on trace les tangentes PA et PB au cercle, A et B étant les points de contact. À un point quelconque M sur la corde AB, on trace une perpendiculaire à OM, qui coupe PA et PB aux points respectifs C et D. Démontrer que M est le milieu du segment CD.

OC222. Soit a et b des nombres naturels tels que $ab > 2$. Sachant que la somme de leur plus grand commun diviseur et de leur plus petit commun multiple est divisible par $a + b$, démontrer que le quotient de cette division est inférieur ou égal à $\frac{a+b}{4}$. Quelles sont les conditions auxquelles le quotient est égal à $\frac{a+b}{4}$?

OC223. Soit $\mathbb{Z}$ l'ensemble des entiers. Déterminer toutes les fonctions $f : \mathbb{Z} \to \mathbb{Z}$ telles que

$$xf(2f(y) - x) + y^2 f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

pour tous $x, y \in \mathbb{Z}$ $(x \neq 0)$.

OC224. Soit un entier n supérieur à 1. Un carré $n \times n$ est divisé en n^2 carrés unités. On veut colorier le grand carré de manière que n carrés unités soient de couleur verte, n carrés unités soient de couleur bleue et les autres soient blancs. Y a-t-il plus de coloriages possibles dans lesquels il y a exactement un carré vert dans chaque rangée et exactement un carré bleu dans chaque colonne ou plus de coloriages possibles dans lesquels il y a exactement un carré vert et exactement un carré bleu dans chaque rangée?

OC225. Déterminer la valeur maximale d'un nombre réel k tel que

$$\frac{a}{1 + 9bc + k(b-c)^2} + \frac{b}{1 + 9ca + k(c-a)^2} + \frac{c}{1 + 9ab + k(a-b)^2} \geq \frac{1}{2}$$

pour tous réels non négatifs $a,\ b,\ c$ qui vérifient l'équation $a + b + c = 1$.

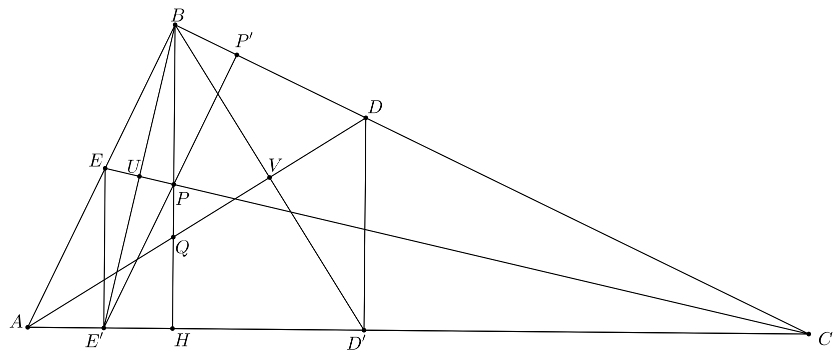
OLYMPIAD SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(3), p. 102–103.

OC161. The altitude BH dropped onto the hypotenuse AC of a triangle ABC intersects the bisectors AD and CE at Q respectively P. Prove that the line passing through the midpoints of segments $[QD]$ and $[PE]$ is parallel to the line AC.

Originally problem 1 from the 2012 Grade IX Romania Math Olympiad

We received ten correct solutions. We present the solution by Michel Bataille.



Let E', D' be the orthogonal projections of E, D onto BC and let P' be the orthogonal projection of P onto BC. The lines CB and CA are symmetric in the bisector CE and it is also the case of PP' and PH (since $P \in CE$ and $PP' \perp CB, PH \perp CA$). It follows that P' and H are symmetric in CE. In a similar way, B and E' are symmetric in CE. We deduce $BP' = HE'$ and consequently

$$PB = \sqrt{PP'^2 + P'B^2} = \sqrt{PH^2 + HE'^2} = PE'.$$

Since we also have $EB = EE'$, the line CE, which passes through P and E, is the perpendicular bisector of the line segment BE'. It follows that CE intersects BE' at its midpoint U. Under the symmetry about U, the image of B is E' and the image of the line BH is the line EE' (since $EE' \parallel BH$). As a result, the image of $P = CE \cap BH$ is $CE \cap EE' = E$, that is, U is also the midpoint of PE. Similarly, the midpoint V of BD' is the midpoint of QD and the desired result follows since the line UV, which passes through the midpoints of BE' and BD' is parallel to $D'E'$.

Editor's Comment. We received a plethora of solutions to this problem with many interesting approaches! In fact, an entire article could probably be written on the number of different viewpoints for this problem.

OC162. Find all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all $k, m, n \in \mathbb{N}$ we have

$$f(km) + f(kn) - f(k)f(mn) \geq 1\,.$$

Originally problem 6 from the 2012 South Africa National Olympiad.

We received two correct solutions and three incorrect submissions. We present the solution by Oliver Geupel.

The constant function $f : x \mapsto 1$ solves the problem. We prove that there is no other solution.

Let f be any solution of the problem and let $P(k, m, n)$ denote the assertion that $f(km) + f(kn) - f(k)f(mn) \geq 1$. By $P(1, 1, 1)$, we have $(f(1) - 1)^2 \leq 0$, whence $f(1) = 1$. From $P(x, 1, 1)$ we see that $f(x) \geq 1$ for every $x \in \mathbb{N}$. It suffices to show that that there is no x such that $f(x) > 1$.

Assume to the contrary that $f(q) > 1$ for some natural number q. From $P(q, 1, x)$ and $P(1, q, x)$, we obtain

$$f(q) + f(qx) - f(q)f(x) \geq 1, \quad (1)$$
$$f(q) + f(x) - f(qx) \geq 1. \quad (2)$$

Adding the inequalities (1) and (2), we obtain $(f(q) - 1) \cdot (2 - f(x)) \geq 0$, thus,

$$f(x) \leq 2 \quad \text{for every } x \in \mathbb{N}. \quad (3)$$

It is enough to show that for every nonnegative integer n it holds

$$f\left(q^{2^n}\right) - 1 \geq f(q)^n \cdot (f(q) - 1). \quad (4)$$

As a consequence, the values $f\left(q^{2^n}\right)$ increase unboundedly, in contradiction to the bound (3).

We prove (4) by induction on n. The base case $n = 0$ is obvious. Assume (4) holds for some $n = k$. By $P\left(q^{2^k}, q^{2^k}, 1\right)$, we have

$$f\left(q^{2^{k+1}}\right) + f\left(q^{2^k}\right) - f\left(q^{2^k}\right)^2 \geq 1.$$

Notice that $f(q^{2k}) \geq f(q)$ using (4) and $(f(q) - 1)(f(q)^k - 1) \geq 0$. By induction, it follows that

$$f\left(q^{2^{k+1}}\right) - 1 \geq f\left(q^{2^k}\right) \cdot \left(f\left(q^{2^k}\right) - 1\right) \geq f(q) \cdot f(q)^k (f(q) - 1) = f(q)^{k+1}(f(q) - 1),$$

which completes the induction and thus the proof.

Editor's Comment. We received two solutions that assumed that the natural numbers begin at 0 (which is not unreasonable). The problem however becomes substantially easier if we allow for this situation, so the editor chose to display a solution without this concern.

OC163. Let $A = \{1, 2, \ldots, 2012\}$, $B = \{1, 2, \ldots, 19\}$ and S be the set of all subsets of A. Find the number of functions $f : S \to B$ such that

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S..$$

Originally problem 1 from day 1 of the 2012 Turkey TST.

We received one solution to this problem. We present Oliver Geupel's solution slightly modified by the editor.

More generally, let $A = \{1, 2, \ldots, m\}$ and $B = \{1, 2, \ldots, n\}$. We prove that the number of functions with the required property is $\sum_{k=1}^{n} k^m$.

For $1 \le k \le n$, let F_k denote the set of all functions $f : S \to B$ with the two properties

$$f(A_1 \cap A_2) = \min\{f(A_1), f(A_2)\} \text{ for all } A_1, A_2 \in S \qquad \text{and} \qquad f(A) = k.$$

Moreover, let G_k denote the set of all functions $g : A \to \{1, 2, \ldots, k\}$.

Note that $|G_k| = k^m$. It is therefore enough to establish bijections

$$F_k \leftrightarrow G_k \qquad (1 \le k \le n).$$

For any $f \in F_k$, let the related $g \in G_k$ be defined by

$$g(j) = f(A \setminus \{j\}) \qquad (1 \le j \le m). \tag{1}$$

Notice that the function is well defined since

$$f(A \setminus \{j\}) = f(A \cap (A \setminus \{j\})) = \min\{f(A), f(A \setminus \{j\}))\} \le k$$

In the following we use the notation $\overline{X} = A \setminus X$. From (1) we deduce by induction on $|\overline{X}|$ that

$$f(X) = \min\left\{g(j) : j \in \overline{X}\right\} \qquad (X \in S \setminus \{A\}),$$

that is, the m relations (1) determine f uniquely.

Now, starting with a $g \in G_k$, pick a $f \in F_k$ such that (1) holds. Then, for any $A_1, A_2 \in S$, we have

$$\begin{aligned} f(A_1 \cap A_2) &= \min\left\{g(j) : j \in \overline{A_1 \cap A_2}\right\} \\ &= \min\left\{g(j) : j \in \overline{A_1} \cup \overline{A_2}\right\} \\ &= \min\left\{\min\left\{g(j) : j \in \overline{A_1}\right\}, \min\left\{g(j) : j \in \overline{A_2}\right\}\right\} \\ &= \min\{f(A_1), f(A_2)\}, \end{aligned}$$

so that $f \in F_k$.

Hence the map $f \mapsto g$ establishes the desired bijection $F_k \leftrightarrow G_k$.

OC164. Find all triples (m, p, q) where m is a positive integer and p, q are primes such that
$$2^m p^2 + 1 = q^5 .$$

Originally problem 3 from day 1 of the 2012 Korean Math Olympiad.

We received five correct solutions and one incorrect submission. We present the solution by Konstantine Zelator.

We will demonstrate that this problem has a unique solution $(m, p, q) = (1, 11, 3)$. Let m, p, q be integers such that $2^m p^2 + 1 = q^5$ where m is a positive integer and p, q are prime numbers. Observe that q must be an odd prime and we can rewrite the equation as

$$2^m p^2 = q^5 - 1 = (q - 1)(q^4 + q^3 + q^2 + q + 1)$$

and note that the factor $q^4 + q^3 + q^2 + q + 1$ is odd. Since

$$q^4 + q^3 + q^2 + q + 1 = (q - 1)(q^3 + 2q^2 + 3q + 4) + 5$$

we see that $\gcd(q-1, q^4+q^3+q^2+q+1) = 1$ or 5. If this greatest common divisor is 5, then the original equation implies that $p = 5$ and hence

$$2^m = \frac{q-1}{5} \cdot \frac{q^4 + q^3 + q^2 + q + 1}{5}.$$

Notice that the factor $\frac{q^4+q^3+q^2+q+1}{5}$ is odd and hence must be 1 since the left hand since has only powers of 2. This would imply that $q = 1$ which is impossible. Thus, the greatest common divisor must be 1.

Now, note that p must also be odd since $q^4 + q^3 + q^2 + q + 1$ is an odd number bigger than 1 and hence has an odd prime divisor. Thus, we must have that

$$q - 1 = 2^m \qquad \text{and} \qquad q^4 + q^3 + q^2 + q + 1 = p^2$$

If $m \geq 3$ then $q \equiv 1 \pmod 8$ and so

$$p^2 = q^4 + q^3 + q^2 + q + 1 \equiv 1^4 + 1^3 + 1^2 + 1 + 1 \equiv 5 \pmod 8$$

which is a contradiction since 5 is not a quadratic residue modulo 8. If $m-2$, then $q = 2^2 + 1 = 5$ and thus

$$p^2 = q^4 + q^3 + q^2 + q + 1 \equiv 5^4 + 5^3 + 5^2 + 5 + 1 \equiv 1 + 5 + 1 + 5 + 1 \equiv 5 \pmod 8$$

which is also a contradiction. Lastly, if $m = 1$, then $q = 3$ and

$$p^2 = 3^4 + 3^3 + 3^2 + 3 + 1 = 121$$

and thus $p = 11$. Hence, we obtain the unique solution to this problem $(m, p, q) = (1, 11, 3)$.

OC165. Let O be the circumcenter of acute ΔABC, and let H be its orthocenter. Let $AD \perp BC$, and let EF be the perpendicular bisector of AO, with D, E on the side BC. Prove that the circumcircle of ΔADE passes through the midpoint of OH.

Originally problem 1 from day 2 of the 2012 China Western Olympiad.

We received six correct solutions. We present the solution by Somasundaram Muralidharan.

Remark. The following solution uses complex slope and complex equations of lines. The basic facts are as follows:

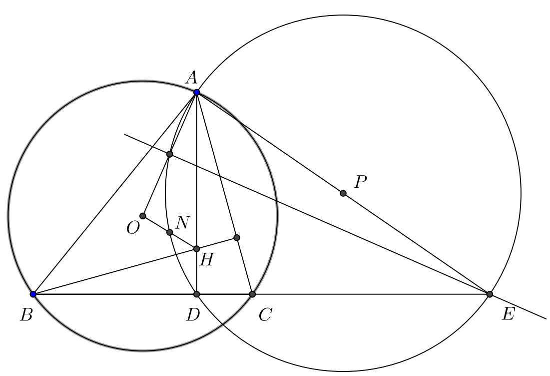
For points $A(z_1)$ and $B(z_2)$, the slope of the line AB is given by $\kappa = \frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}}$. If the line makes an angle θ with the positive real axis, then the slope of the line is the complex number $e^{2i\theta}$.

The equation of the line AB is given by $z - z_1 = \kappa(\bar{z} - \bar{z_1})$. Two lines with slopes κ_1 and κ_2 are parallel if and only if $\kappa_1 = \kappa_2$ and they are perpendicular if and only if $\kappa_1 + \kappa_2 = 0$.

If $A(z_1)$ and $B(z_2)$ lie on the unit circle, $|z| = 1$, then the slope of the line AB is given by

$$\frac{z_1 - z_2}{\bar{z_1} - \bar{z_2}} = \frac{z_1 - z_2}{\frac{1}{z_1} - \frac{1}{z_2}} = -z_1 z_2.$$

For more details, see *P.S.Modenov, Problems in Geometry, MIR Publishers, 1981.*



There is no loss of generality in assuming that the circumcenter O is at the origin and that the circumradius is 1. Let the complex numbers representing the vertices A, B, C be respectively a, b, c. Then the orthocenter H is represented by the complex number $a + b + c$ and N, the mid point of OH is the complex number $\frac{a+b+c}{2}$.

For any point R, let $\bar{R}$ represent the reflection of the point R on the real axis. If z is the complex number representing R, then the complex number representing

$\bar{R}$ will be the complex conjugate of z, represented by $\bar{z}$.

The slope of the line OA is $\dfrac{a}{\bar{a}} = a^2$. The equation of the perpendicular bisector of OA is

$$Z - \frac{a}{2} = -a^2\left(\bar{Z} - \frac{\bar{a}}{2}\right). \tag{1}$$

Simplifying (1), we obtain the equation of the perpendicular bisector of OA as

$$Z - a = -a^2\bar{Z}. \tag{2}$$

The equation of the line BC is

$$Z - b = -bc(\bar{Z} - \bar{b})$$

which can be written as

$$Z - b - c = -bc\bar{Z}. \tag{3}$$

We can assume that OA is not perpendicular to BC, since, in this case, the above two lines are parallel. Since the slopes are $-a^2$ and $-bc$, the condition that these two intersect is $-a^2 \neq -bc$ or $a^2 - bc \neq 0$.

Solving (2) and (3) we can obtain the point E. From (2) and (3), we get

$$\bar{Z} = \frac{a - b - c}{a^2 - bc}.$$

This represents the point $\bar{E}$, the reflection of E on the real axis. If P is the center of the circle ADE, then P is the mid point of AE (since $\angle ADE = 90^\circ$). The point $\bar{A}$ is given by $\bar{a}$ and hence $\bar{P}$ is given by

$$\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} + \bar{a}\right).$$

Thus the radius of the circle ADE is

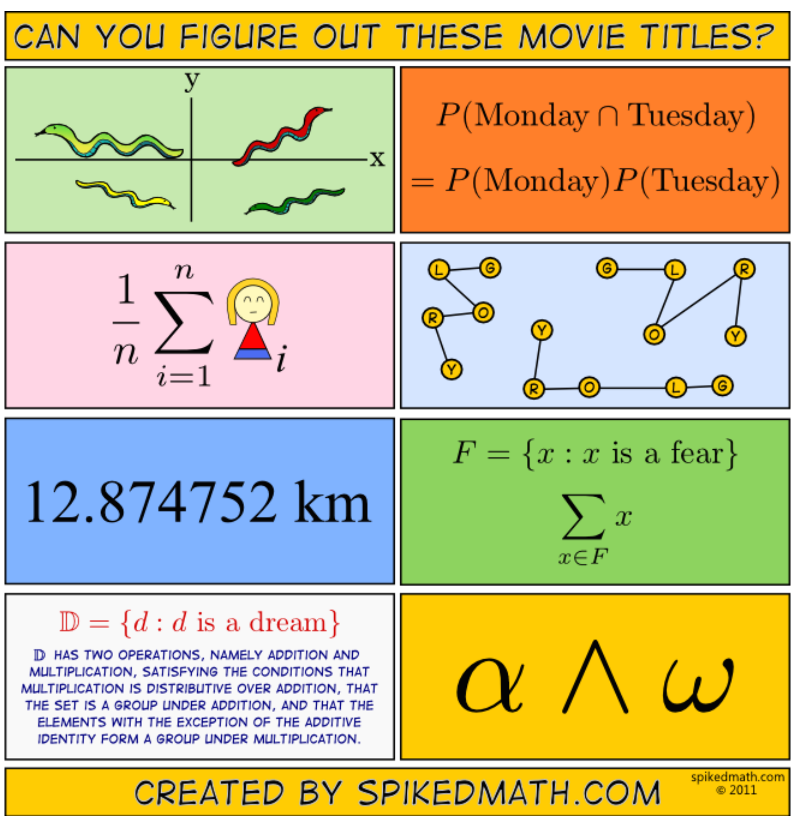
$$\begin{aligned}\rho = |AP| = |\bar{A}\bar{P}| &= \left|\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} + \bar{a}\right) - \bar{a}\right| \\ &= \left|\frac{1}{2}\left(\frac{a - b - c}{a^2 - bc} - \bar{a}\right)\right| \\ &= \frac{1}{2}\left|\frac{\bar{a}bc - b - c}{a^2 - bc}\right|.\end{aligned}$$

Now, it suffices to show that the distance between $\bar{N}$ and $\bar{P}$ is also equal to ρ.

Then,

$$\begin{aligned}
|\bar{N}\bar{P}| &= \left|\frac{1}{2}\left(\frac{a-b-c}{a^2-bc}+\bar{a}\right)-\left(\frac{\bar{a}+\bar{b}+\bar{c}}{2}\right)\right| \\
&= \frac{1}{2}\left|\frac{a-b-c}{a^2-bc}-\bar{b}-\bar{c}\right| \\
&= \frac{1}{2}\left|\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \\
&= \frac{1}{2}|\bar{a}^2bc|\left|\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \qquad \text{since } |a|=|b|=|c|=1 \\
&= \frac{1}{2}\left|(\bar{a}^2bc)\frac{a-a^2\bar{b}-a^2\bar{c}}{a^2-bc}\right| \\
&= \frac{1}{2}\left|\frac{\bar{a}bc-c-b}{a^2-bc}\right|.
\end{aligned}$$

Hence $|\bar{N}\bar{P}| = \rho$ and the circle ADE passes through N, the midpoint of OH.



FOCUS ON...

No. 16

Michel Bataille

Leibniz's and Stewart's relations

Introduction

Let $A_1, A_2, \ldots, A_n$ be points in space and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be real numbers. A quick study of the sum

$$\ell(M) = \alpha_1 MA_1^2 + \alpha_2 MA_2^2 + \cdots + \alpha_n MA_n^2$$

where M is an arbitrary point, will lead to Leibniz's relation, a result often proved as a lemma when needed and which certainly deserves to be better known. As a by-product, we will obtain Stewart's relation, naturally linked to this study.

Transformation of $\ell(M)$

We introduce the vectorial version of ℓ defined by

$$\overrightarrow{\mathcal{L}}(M) = \alpha_1 \overrightarrow{MA_1} + \alpha_2 \overrightarrow{MA_2} + \cdots + \alpha_n \overrightarrow{MA_n}$$

which obviously satisfies $\overrightarrow{\mathcal{L}}(M) - \overrightarrow{\mathcal{L}}(N) = \alpha \overrightarrow{MN}$ where $\alpha = \sum_{k=1}^{n} \alpha_k$ for all M, N. From this relation, it follows that $\overrightarrow{\mathcal{L}}(M)$ is a vector $\overrightarrow{U}$ independent of M when $\alpha = 0$ and otherwise that $\overrightarrow{\mathcal{L}}(G) = \overrightarrow{0}$ for a unique point G, the centre of mass of the weighted points (A_k, α_k), $k = 1, 2, \ldots, n$. Note that in the latter case, $\overrightarrow{\mathcal{L}}(M) = \alpha \overrightarrow{MG}$ for any point M.

Now, using $MA^2 - NA^2 = (\overrightarrow{MA} - \overrightarrow{NA}) \cdot (\overrightarrow{MA} + \overrightarrow{NA})$, we readily obtain

$$\ell(M) - \ell(N) = \overrightarrow{MN} \cdot (\overrightarrow{\mathcal{L}}(M) + \overrightarrow{\mathcal{L}}(N)).$$

This gives $\ell(M) - \ell(N) = 2\overrightarrow{MN} \cdot \overrightarrow{U}$ when $\alpha = 0$ and, if $\alpha \neq 0$, what is generally called Leibniz's relation: $\ell(M) = \ell(G) + \overrightarrow{MG} \cdot \overrightarrow{\mathcal{L}}(M) = \ell(G) + \alpha MG^2$, that is,

$$\alpha_1 MA_1^2 + \alpha_2 MA_2^2 + \cdots + \alpha_n MA_n^2 = \alpha MG^2 + \alpha_1 GA_1^2 + \alpha_2 GA_2^2 + \cdots + \alpha_n GA_n^2.$$

To see these formulas at work in elementary examples, consider two distinct points B, C. Denoting by I the midpoint of BC and by A an arbitrary point, Leibniz's relation gives $AB^2 + AC^2 = 2AI^2 + IA^2 + IB^2$ leading to $4AI^2 = 2AB^2 + 2AC^2 - BC^2$, the familiar formula for the median AI in triangle ABC. On the other hand, consider the equality $MB^2 - MC^2 = AB^2 - AC^2$ for some point M. This relation rewrites as $(MB^2 - MC^2) - (AB^2 - AC^2) = 0$ or $2\overrightarrow{MA} \cdot \overrightarrow{CB} = 0$ (here $\alpha = 0$ and

$\overrightarrow{U} = \overrightarrow{CB}$). Thus, the locus of the points M such that $MB^2 - MC^2 = AB^2 - AC^2$ is the plane through A orthogonal to BC.

Two applications

As a first application, we consider problem 11433 of *The American Mathematical Monthly* (slightly reformulated):

> Let n be a positive integer, and $A_1, \ldots, A_n, B_1, \ldots, B_n$, and $C_1, \ldots, C_n$ be points on the unit sphere S^2. Show that there exists P on S^2 such that
> $$\sum_{k=1}^{n} PA_k^2 = \sum_{k=1}^{n} PB_k^2 = \sum_{k=1}^{n} PC_k^2.$$

Call E the centre of mass of $(A_1, 1), \ldots, (A_n, 1)$. From Leibniz's relation, we have

$$\sum_{k=1}^{n} MA_k^2 = nME^2 + \sum_{k=1}^{n} EA_k^2$$

for any point M in space. Taking M at O, the centre of S^2, we obtain

$$\sum_{k=1}^{n} EA_k^2 = n(1 - OE^2)$$

and so

$$\sum_{k=1}^{n} MA_k^2 = n(1 + ME^2 - OE^2).$$

Similarly, if F is the centre of mass of $(B_1, 1), \ldots, (B_n, 1)$ and G the one of $(C_1, 1), \ldots, (C_n, 1)$, the following equalities hold:

$$\sum_{k=1}^{n} MB_k^2 = n(1 + MF^2 - OF^2), \qquad \sum_{k=1}^{n} MC_k^2 = n(1 + MG^2 - OG^2).$$

Now, let $\mathcal{P}_1 = \{M : ME^2 - MF^2 = OE^2 - OF^2\}$ and $\mathcal{P}_2 = \{M : MF^2 - MG^2 = OF^2 - OG^2\}$. We observe that a suitable point P is a point of S^2 which also belongs to $\mathcal{P}_1 \cap \mathcal{P}_2$. The existence of such a point P is ensured by the following discussion.

• If $E = F = G$, every point of S^2 is suitable.

• If, say, $E \neq F$ and $F = G$, then $\mathcal{P}_1 = \mathcal{P}_1 \cap \mathcal{P}_2$ is the plane through O orthogonal to the line EF. Any point of the great circle intersection of this plane with S^2 is suitable. The same conclusion holds if E, F, G are distinct and collinear since then $\mathcal{P}_1 = \mathcal{P}_2$.

• In the general case where E, F, G are not collinear, $\mathcal{P}_1$ and $\mathcal{P}_2$ are distinct planes which both pass through O. They intersect along a diameter of S^2 and this diameter intersects S^2 in two suitable points.

Our second example is provided by Christopher Bradley's problem **2629** [2001 : 214 ; 2002 : 256]:

In triangle ABC, the symmedian point is denoted by S. Prove that

$$\frac{1}{3}(AS^2 + BS^2 + CS^2) \geq \frac{BC^2AS^2 + CA^2BS^2 + AB^2CS^2}{BC^2 + CA^2 + AB^2}.$$

We propose the following variant of Joel Schlosberg's featured solution.

Let $BC = a, CA = b, AB = c$ and let G be the centroid of ΔABC. Since S is the centre of mass of $(A, a^2), (B, b^2), (C, c^2)$ and G is the centre of mass of $(A, 1), (B, 1), (C, 1)$, Leibniz's relation yields on the one hand

$$a^2GA^2 + b^2GB^2 + c^2GC^2 = (a^2 + b^2 + c^2)GS^2 + a^2SA^2 + b^2SB^2 + c^2SC^2$$

and on the other hand,

$$AS^2 + BS^2 + CS^2 = 3GS^2 + AG^2 + BG^2 + CG^2.$$

It easily follows that the required inequality is equivalent to

$$6(a^2+b^2+c^2)GS^2 \geq (2a^2-b^2-c^2)GA^2+(2b^2-c^2-a^2)GB^2+(2c^2-a^2-b^2)GC^2 \quad (1)$$

Using $GA^2 = \frac{1}{9}(2b^2 + 2c^2 - a^2)$ *etc.*, the right-hand side R of (1) becomes after some calculations, $R = -\frac{1}{3}((a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2)$, hence $R \leq 0$ and (1) follows.

Note that equality holds if and only if the triangle ABC is equilateral.

Stewart's relation

Equality (2) below, generally called Stewart's relation, relates four points, three of which are collinear.

Let A, B, C be points on a line δ and let M be an arbitrary point. Then

$$\overline{BC}.MA^2 + \overline{CA}.MB^2 + \overline{AB}.MC^2 + \overline{BC}.\overline{CA}.\overline{AB} = 0 \qquad (2)$$

where $\overline{XY}$ denotes the signed distance from X to Y.

The proof is a nice application of the case $\alpha = 0$ in the general study of $\ell(M)$.

Let $\ell(M) = \overline{BC}.MA^2 + \overline{CA}.MB^2 + \overline{AB}.MC^2$. Since $\overline{BC} + \overline{CA} + \overline{AB} = 0$, we have $\ell(M) - \ell(N) = 2\overrightarrow{MN} \cdot \overrightarrow{U}$ where $\overrightarrow{U}$ is the corresponding constant vector

$$\overrightarrow{U} = \overrightarrow{\mathcal{L}}(M) = \overline{BC}\,\overrightarrow{MA} + \overline{CA}\,\overrightarrow{MB} + \overline{AB}\,\overrightarrow{MC} = \overline{CA}\,\overrightarrow{AB} + \overline{AB}\,\overrightarrow{AC} = \overrightarrow{0}.$$

Thus, $\ell(M)$ is independent of M as well, and so $\ell(M) = \ell(A) = \overline{CA}.\overline{AB}.\overline{CB}$, as desired.

The application coming to mind at once is to the cevians of a triangle:

Let $D = tB + (1-t)C$ be a point of the sideline BC of triangle ABC ($t \in \mathbb{R}$). Then, the length of the cevian AD is given by

$$AD^2 = tc^2 + (1-t)b^2 - t(1-t)a^2 \qquad (3)$$

where, as usual, $a = BC, b = CA, c = AB$.

This directly follows from (2) with $M = A$ and points B, D, C on line BC. Indeed, we first get

$$\overline{DC}.AB^2 + \overline{BD}.AC^2 + \overline{CB}.AD^2 + \overline{DC}.\overline{BD}.\overline{CB} = 0$$

and then, since $\overline{BD} = (1-t)\overline{BC},\ \overline{DC} = t\overline{BC}$,

$$t\overline{BC}.c^2 + (1-t)\overline{BC}.b^2 - \overline{BC}.AD^2 - t\overline{BC}.(1-t)\overline{BC}.\overline{BC} = 0.$$

Dividing by $\overline{BC}$ yields (3).

In particular, if AD is the internal bisector of $\angle BAC$, then $t = \dfrac{b}{b+c}$ and so

$$AD^2 = \frac{b}{b+c} \cdot c^2 + \frac{c}{b+c} \cdot b^2 - a^2 \cdot \frac{bc}{(b+c)^2}.$$

This leads to the known formula $AD^2 = bc - \dfrac{a^2bc}{(b+c)^2}$.

For another particular case, consider the symmedian point S that we met earlier. If AS intersects BC at D, we obtain the length of the symmedian from A:

$$AD = \frac{2bcm_a}{b^2+c^2}$$

where m_a is the length of the median from A; this follows from (3) with $t = \dfrac{b^2}{b^2+c^2}$.

We conclude with two exercises.

Exercises

1. Let P be an arbitrary point in the plane of a triangle ABC with sidelengths a, b, c. Prove that

$$PA^2 + PB^2 + PC^2 \geq \frac{a^2+b^2+c^2}{3}.$$

2. Let A, B, C, D be four points on a line ℓ in this order and let M not on ℓ be such that $\angle AMB = \angle CMD$. Prove that

$$\frac{MA^2}{MC^2} > \frac{AB}{CD} > \frac{MB^2}{MD^2}.$$

Application of Inversive Methods to Euclidean Geometry

Andy Liu

Place a sphere on top of the Euclidean plane so that its south pole S is at the origin. Let N be the north pole. For any point $Q \neq N$ on the sphere, the point P of intersection of the extension of NQ with the plane is called its image under the **stereographic projection** from N.

Of course, it would be tidier if N had an image as well. How would it behave? As Q approaches N from any direction, the projection P "approaches infinity" in the sense of becoming arbitrarily far away from S. If we add a *point at infinity* I to the Euclidean plane, with the property that a sequence (P_j) is defined to converge to I if and only if $|P_j|$ increases without bound, we will have what is known as the **inversive plane**. Think of the sphere as a balloon and the point N as a puncture. If we stretch the balloon out onto the plane, we can see that the point I is in every direction!

We define the point I to be the projection of N. It is called the **ideal point**, and lies on every straight line. To see this, consider a straight line ℓ on the inversive plane and the plane passing through N and ℓ. The cross-section with the sphere is a circle passing through the point N, justifying the statement that I lies on every straight line. In fact, it closes the straight line into something like a circle.

Inversion

For any circle Σ with center O and radius R, and any point $A \neq O, I$, we define the *inverse point of A with respect to* Σ to be the point on the ray $\overrightarrow{OA}$ at distance $R^2/|OA|$ from O. This is readily seen to be an involution (self-inverse map). The points O and I are defined to invert into each other. We consider straight lines to be "circles passing through I". Inversion in a straight line is defined to be reflection: the point I is fixed under reflections. The geometry resulting from (and preserved by) these mappings is called *inversive geometry.* For a full introduction to inversive geometry, the reader is referred to any good undergraduate geometry textbook, such as Pedoe [1] (chapter VI) or Baragar [2] (chapter 7).

Exercise 1 *Inversion fixes exactly the points of Σ. It maps points inside Σ to points outside Σ and* vice versa.

The next result is a very useful lemma. Note the order in which the points of the triangles are specified - this is important!

Exercise 2 *Let P, Q, and the center of inversion O not be collinear, and let P,Q invert to P',Q'. Then the triangles $\triangle OPQ$ and $\triangle OQ'P'$ are similar.*

The reflection of a circle in a line is always a circle. Something similar is true for inversions.

Exercise 3 *Inversion maps circles not passing through O to circles, circles passing through O to straight lines not through O, straight lines passing through O to themselves, and other straight lines to circles.*

We can define the angle between two circles, or between a circle and a line, at a point P to be the angle between the tangent lines. Reflection preserves these, of course — so does inversion.

Exercise 4 *Show that inversion preserves angles, whether between two lines, a line and a circle, or two circles.*

Reflection in a line L maps any line or circle that is orthogonal to L to itself. (Note that if a circle meets a line or another circle twice, it makes the same angle at each intersection point. Thus "orthogonal" is well defined here and in the following exercise.)

Exercise 5 *Show that inversion maps any circle orthogonal to Σ to itself.*

Exercise 6 *If a circle C cuts Σ, so does its inverse. If a circle C is tangent to Σ, so is its inverse. If a circle C contains O in its interior, so does its inverse.*

Exercise 7 *The Euclidean construction for an inverse point is simple enough to find by trial and error.*

(i) Given O and Σ, and a point P inside Σ, construct the inverse point P'.

(ii) Given O and Σ, and a point P outside Σ, construct the inverse point P'.

Reflection preserves reflections: that is, a mirror seen in a mirror acts like a mirror. Something similar holds for inversions:

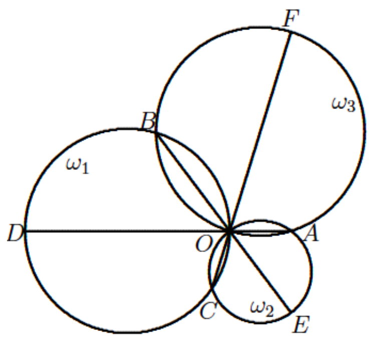
Exercise 8 *If P and P' are inverse with respect to C, and their inverses with respect to Σ are $\overline{P}$,$\overline{P'}$, and $\overline{C}$ respectively, then $\overline{P}$ and $\overline{P'}$ are inverses with respect to $\overline{C}$.*

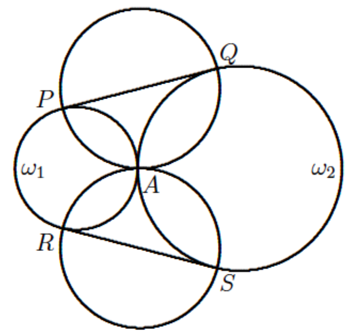
We might wonder if inversion preserves circle centers, but it doesn't. (It's easy to find a counterexample — find one!) There is a way to find the center of an inverse circle, though.

Exercise 9 *If C and $\overline{C}$ are inverses with respect to Σ, then the center A of $\overline{C}$ is found as follows. Let B be the inverse of O in C; then A is the inverse of B in Σ.*

Problems

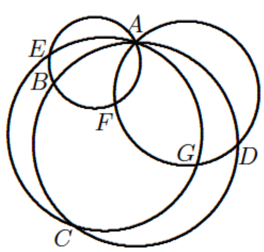
Problem 1 (below left)
Three circles ω_1, ω_2 and ω_3 pass through O. C is the other point of intersection of ω_1 and ω_2, A is the other point of intersection of ω_2 and ω_3, and B is the other point of intersection of ω_3 and ω_1. The extension of AO intersects ω_1 again at D, the extension of BO intersects ω_2 again at E, and the extension of CO intersects ω_3 again at F. Prove that if OE and OF are diameters of ω_2 and ω_3 respectively, then OD is a diameter of ω_1.

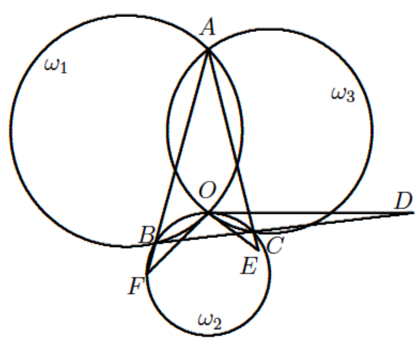




Problem 2 (above right)
Two circles ω_1 and ω_2 are tangent externally to each other at A. A common exterior tangent touches ω_1 at P and ω_2 at Q. The other common exterior tangent touches ω_1 at R and ω_2 at S. Prove that the circumcircles of triangles PAQ and RAS are tangent to each other.

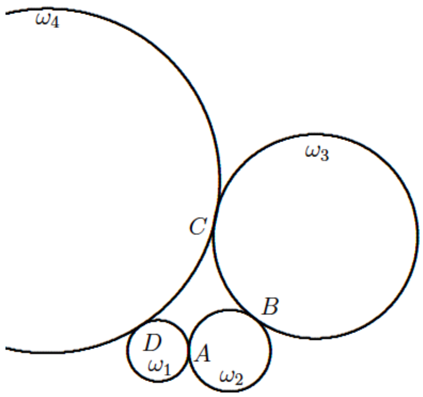
Problem 3 (below left)
AB, AC and AD are three chords on a circle. Circles with AB and AC as diameters intersect at E, circles with AB and AD as diameters intersect at F, and circles with diameters AC and AD intersect at G. Prove that E, F and G are collinear.

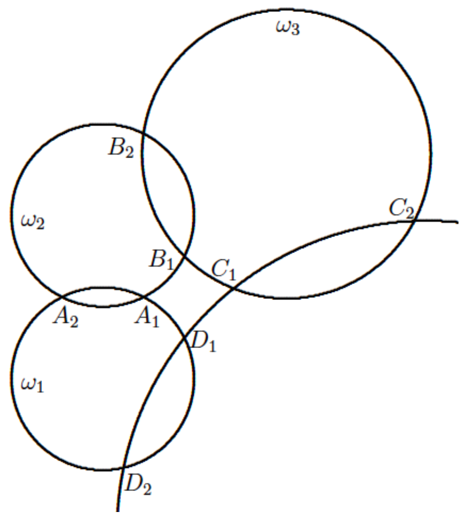




Problem 4 (above right)
Three circles ω_1, ω_2 and ω_3 pass through O. B is the other point of intersection of ω_1 and ω_2, C is the other point of intersection of ω_2 and ω_3, and A is the other point of intersection of ω_3 and ω_1. The tangent to ω_2 at O intersects BC at D, the tangent at O to ω_3 intersects CA at E, and the tangent at O to ω_1 intersects AB at F. Prove that D, E and F are collinear.

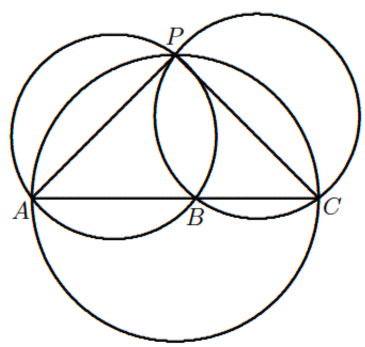
Problem 5 (below left)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 touch at A, ω_2 and ω_3 touch at B, ω_3 and ω_4 touch at C and ω_4 and ω_1 touch at D. Prove that A, B, C and D are concyclic.





Problem 6 (above right)
Four circles ω_1, ω_2, ω_3 and ω_4 are such that ω_1 and ω_2 intersect at A_1 and A_2, ω_2 and ω_3 intersect at B_1 and B_2, ω_3 and ω_4 intersect at C_1 and C_2, and ω_4 and ω_1 intersect at D_1 and D_2. Prove that if A_1, B_1, C_1 and D_1 are collinear or concyclic, then so are A_2, B_2, C_2 and D_2.

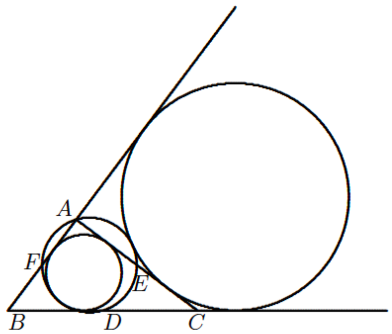
Problem 7 (below)
A, B and C are three points on a line and P is a point not on this line. Prove that the circumcentres of triangles PAB, PBC and PCA are concyclic with P.

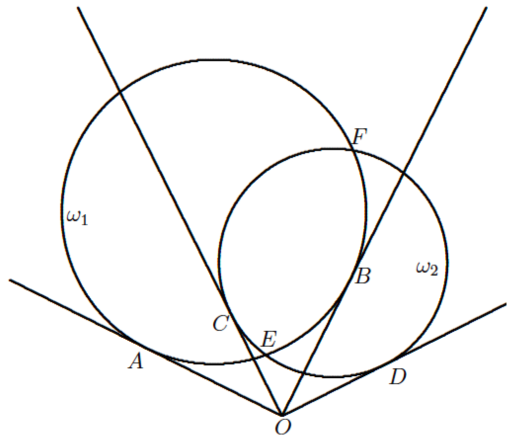


Problem 8
Prove Ptolemy's Inequality which states that $AB \cdot CD + AD \cdot BC \geq AC \cdot BD$ for any convex quadrilateral $ABCD$, with equality if and only if the quadrilateral is cyclic. (Hint: Because this is quantitative, expect to use the "polar-coordinate" definition of inversion.)

Problem 9 (below left)
Prove that the circle which passes through the midpoints of the sides of a triangle

is tangent to the triangle's incircle and excircles.

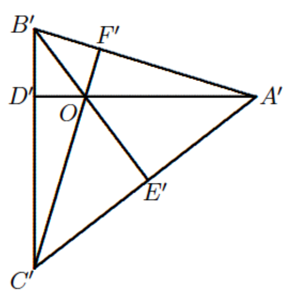




Problem 10 (above right)
From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.

The solution to Problem 1 is given as an example. We leave the others to the reader!

Solution (to Problem 1) Invert with respect to any circle with center O. Then the three circles turn into triangle $A'B'C'$ while the radial lines OA, OB and OC invert to themselves. That OE is a diameter of ω_2 means that $B'E'$ is orthogonal to $A'C'$. Similarly, $C'F'$ is orthogonal to $A'B'$. Hence O is the orthocentre of triangle $A'B'C'$, so that $A'O$ is orthogonal to $B'C'$. It follows that OD is indeed a diameter of ω_1.



References

[1] Baragar, A., *A Survey of Classical and Modern Geometries*, Pearson, 2001.

[2] Pedoe, D., *Geometry: A Comprehensive Course*, Dover, 1970.

PROBLEMS

Readers are invited to submit solutions, comments and generalizations to any problem in this section. Moreover, readers are encouraged to submit problem proposals. Please see submission guidelines inside the back cover or online.

To facilitate their consideration, solutions should be received by the editor by **May 1, 2016**, *although late solutions will also be considered until a solution is published.*

The editor thanks André Ladouceur, Ottawa, ON, for translations of the problems.

An asterisk ($\star$) *after a number indicates that a problem was proposed without a solution.*

4021. *Proposed by Arkady Alt.*

Let $(\overline{\mathbf{a}}_n)_{n\geq 0}$ be a sequence of Fibonacci vectors defined recursively by $\overline{\mathbf{a}}_0 = \overline{\mathbf{a}}, \overline{\mathbf{a}}_1 = \overline{\mathbf{b}}$ and $\overline{\mathbf{a}}_{n+1} = \overline{\mathbf{a}}_n + \overline{\mathbf{a}}_{n-1}$ for all integers $n \geq 1$. Prove that, for all integers $n \geq 1$, the sum of vectors $\overline{\mathbf{a}}_0 + \overline{\mathbf{a}}_1 + \cdots + \overline{\mathbf{a}}_{4n+1}$ equals $k\overline{\mathbf{a}}_i$ for some i and constant k.

4022. *Proposed by Leonard Giugiuc.*

In a triangle ABC, let internal angle bisectors from angles A, B and C intersect the sides BC, CA and AB in points D, E and F and let the incircle of ΔABC touch the sides in M, N, and P, respectively. Show that

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

4023. *Proposed by Ali Behrouz.*

Find all functions $f : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}$ with $x > y$, we have

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x)).$$

4024. *Proposed by Leonard Giugiuc.*

Let a, b, c and d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$abc + abd + acd + bcd + 4 \geq a + b + c + d$$

and determine when equality holds.

4025. *Proposed by Dragoljub Miloşević.*

Prove that for positive numbers a, b and c, we have

$$\sqrt[3]{\left(\frac{a}{2b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{2c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{2a+b}\right)^2} \geq \sqrt[3]{3}.$$

4026. *Proposed by Roy Barbara.*

Prove or disprove the following property: if r is any non-zero rational number, then the real number $x = (1 + r)^{1/3} + (1 - r)^{1/3}$ is irrational.

4027. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{ab}{a + ab + b} + \frac{bc}{b + bc + c} + \frac{ac}{a + ac + c} \leq 1.$$

4028. *Proposed by Michel Bataille.*

In 3-dimensional Euclidean space, a line ℓ meets orthogonally two distinct parallel planes $\mathcal{P}$ and $\mathcal{P}'$ at H and H'. Let r and r' be positive real numbers with $r \leq r'$; let $\mathcal{C}$ be the circle in $\mathcal{P}$ with center H, radius r, and let $\mathcal{C}'$ in $\mathcal{P}'$ be similarly defined. For a fixed point M' on $\mathcal{C}'$, find the maximum distance between the lines ℓ and MM' as M moves about the circle $\mathcal{C}$ (where the distance between two lines is the minimum distance from a point of one line to a point of the other).

4029. *Proposed by Paul Bracken.*

Suppose $a > 0$. Find the solutions of the following equation in the interval $(0, \infty)$:

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a.$$

4030. *Proposed by Paolo Perfetti.*

a) Prove that $4^{\cos t} + 4^{\sin t} \geq 5$ for $t \in [0, \frac{\pi}{4}]$.

b) Prove that $6^{\cos t} + 6^{\sin t} \geq 7$ for $t \in [0, \frac{\pi}{4}]$.

. . . .

4021. *Proposé par Arkady Alt.*

Soit $(\overline{\mathbf{a}}_n)_{n\geq 0}$ une suite de vecteurs définie de façon récursive à la manière de Fibonacci : $\overline{\mathbf{a}}_0 = \overline{\mathbf{a}}, \overline{\mathbf{a}}_1 = \overline{\mathbf{b}}$ et $\overline{\mathbf{a}}_{n+1} = \overline{\mathbf{a}}_n + \overline{\mathbf{a}}_{n-1}$. Démontrer que la somme $\overline{\mathbf{a}}_0 + \overline{\mathbf{a}}_1 + \cdot + \overline{\mathbf{a}}_{4n+1}$ est égale à $k\overline{\mathbf{a}}_i$ pour un i quelconque et une constante k.

4022. *Proposé par Leonard Giugiuc.*

Dans un triangle ABC, les bissectrices internes des angles A, B et C coupent les côtés BC, CA et AB aux points respectifs D, E et F. De plus, le cercle inscrit dans le triangle touche ces mêmes côtés aux points respectifs M, N et P. Démontrer que

$$\frac{PA}{PB} + \frac{MB}{MC} + \frac{NC}{NA} \geq \frac{FA}{FB} + \frac{DB}{DC} + \frac{EC}{EA}.$$

4023. *Proposé par Ali Behrouz.*

Déterminer toutes les fonctions f ($f : \mathbb{R}^+ \mapsto \mathbb{R}^+$) pour lesquelles

$$f\left(\frac{x}{x-y}\right) + f(xf(y)) = f(xf(x))$$

pour tous réels x et y ($x > y$).

4024. *Proposé par Leonard Giugiuc.*

Soit a, b, c et d des réels tels que $a^2 + b^2 + c^2 + d^2 = 4$. Démontrer que

$$abc + abd + acd + bcd + 4 \geq a + b + c + d$$

et déterminer les conditions auxquelles il y a égalité.

4025. *Proposé par Dragoljub Miloşević.*

Soit a, b et c des réels strictement positifs. Démontrer que

$$\sqrt[3]{\left(\frac{a}{2b+c}\right)^2} + \sqrt[3]{\left(\frac{b}{2c+a}\right)^2} + \sqrt[3]{\left(\frac{c}{2a+b}\right)^2} \geq \sqrt[3]{3}.$$

4026. *Proposé par Roy Barbara.*

Démontrer ou infirmer l'énoncé suivant: Si r est un nombre rationnel non nul, alors le nombre x défini par $x = (1+r)^{1/3} + (1-r)^{1/3}$ est irrationnel.

4027. *Proposé par George Apostolopoulos.*

Soit a, b et c des réels strictement positifs tels que $a + b + c = 3$. Démontrer que

$$\frac{ab}{a+ab+b} + \frac{bc}{b+bc+c} + \frac{ac}{a+ac+c} \leq 1.$$

4028. *Proposé par Michel Bataille.*

Dans l'espace euclidien à trois dimensions, on considère une droite ℓ orthogonale à deux plans parallèles distincts, $\mathcal{P}$ et $\mathcal{P}'$, qui coupe ces plans aux points respectifs H et H'. Soit r et r' des réels strictement positifs tels que $r \leq r'$. Soit $\mathcal{C}$ le cercle dans $\mathcal{P}$ de centre H et de rayon r. De même, $\mathcal{C}'$ est le cercle dans $\mathcal{P}'$ de centre

H' et de rayon r'. Étant donné un point fixe M' sur $\mathcal{C}'$, déterminer la distance maximale entre les droites ℓ et MM' lorsque M se meut autour du cercle $\mathcal{C}$. (La distance entre deux droites est la distance minimale entre un point d'une droite et un point de l'autre droite.)

4029. *Proposé par Paul Bracken.*

Soit un réel a ($a > 0$). Résoudre l'équation suivante dans l'intervalle $(0, \infty)$:

$$\frac{1}{x+1} + \sum_{n=1}^{\infty} \frac{n!}{(x+1)(x+2)\cdots(x+n+1)} = x - a$$

4030. *Proposé par Paolo Perfetti.*

a) Démontrer que $4^{\cos t} + 4^{\sin t} \geq 5$, pour tout t dans l'intervalle $[0, \frac{\pi}{4}]$.

b) Démontrer que $6^{\cos t} + 6^{\sin t} \geq 7$, pour tout t dans l'intervalle $[0, \frac{\pi}{4}]$.

Math Quotes

Numbers written on restaurant bills within the confines of restaurants do not follow the same mathematical laws as numbers written on any other pieces of paper in any other parts of the Universe.

This single statement took the scientific world by storm. It completely revolutionized it. So many mathematical conferences got held in such good restaurants that many of the finest minds of a generation died of obesity and heart failure and the science of math was put back by years.

Douglas Adams, "Life, the Universe and Everything." New York: Harmony Books, 1982.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2014: 40(3), p. 120–124.

3921. *Proposed by Michel Bataille.*

Let $AUVW$ be a rectangle with $UV = 1$. In each of the following cases (a) $VW = 8$ and (b) $VW = 9$, is it possible to construct with ruler and compass points B on ray $[AU)$ and C on ray $[AW)$ such that V and the orthogonal projection of A onto BC are symmetric about the midpoint of BC?

We received six correct submissions. We present the solution by Titu Zvonaru.

With D denoting the orthogonal projection of A onto BC, the constructed figure will consist of a right triangle ABC with points V and D on the hypotenuse BC so that $BV = CD$. Denoting $a = AU$ and $x = UB$, we obtain by similitude ($\Delta ABC \sim \Delta UBV \sim \Delta DAC$),

$$AC = \frac{x+a}{x} \quad \text{and} \quad CD = \frac{AC^2}{BC}.$$

Consequently,

$$BC^2 = (x+a)^2 + \frac{(x+a)^2}{x^2} = \frac{(x+a)^2(x^2+1)}{x^2} \quad \text{and} \quad BV^2 = x^2+1.$$

Now, $BV^2 = CD^2$ then implies that

$$x^2+1 = \frac{(x+a)^4}{x^4} \cdot \frac{x^2}{(x+a)^2(x^2+1)}, \quad \text{or} \quad (x^2+1)^2 = \left(\frac{x+a}{x}\right)^2,$$

which (because all lengths must be positive) reduces to $x = \sqrt[3]{a}$.

Case a). If $a = 8$, then $x = 2$ and *yes*, we can construct the point B with ruler and compass so that $UB = 2 = 2UV$ (and then C is the point where BV intersects AW).

Case b). If $a = 9$, then we *cannot* construct with ruler and compass the required points B and C. (A segment whose length is the cube root of a rational number cannot be constructed unless it is rational, but $\sqrt[3]{9}$ is not rational.)

3922. *Proposed by Marcel Chiriţă.*

Let M be a point inside a triangle ABC. Show that

$$\frac{(x+y+z)^9}{729xyz} \geq a^2b^2c^2,$$

where $MA = x, MB = y, MC = z$ and $BC = a, AC = b, AB = c$.

We received four correct solutions and one comment. We present the solution by AN-anduud Problem Solving Group.

Observe that

$$\left(\frac{1}{x}\overrightarrow{MA} + \frac{1}{y}\overrightarrow{MB} + \frac{1}{z}\overrightarrow{MC}\right) \cdot \left(\frac{1}{x}\overrightarrow{MA} + \frac{1}{y}\overrightarrow{MB} + \frac{1}{z}\overrightarrow{MC}\right) \geq 0.$$

Since $\overrightarrow{MA} \cdot \overrightarrow{MB} = xy(\cos \angle AMB) = (1/2)(x^2 + y^2 - c^2)$, with similar expressions for other dot products, we obtain that

$$3 + \frac{1}{xy}(x^2 + y^2 - c^2) + \frac{1}{yz}(y^2 + z^2 - a^2) + \frac{1}{zx}(z^2 + x^2 - b^2) \geq 0.$$

Multiplying by xyz leads to

$$(xy + yz + zx)(x + y + z) \geq xa^2 + yb^2 + zc^2.$$

Since $x^2 + y^2 + z^2 \geq xy + yz + zx$, then $(x + y + z)^2 \geq 3(xy + yz + zx)$, and

$$\frac{(x + y + z)^3}{3} \geq (xy + yz + zx)(x + y + z) \geq xa^2 + yb^2 + zc^2 \geq 3\sqrt[3]{a^2b^2c^2xyz}.$$

Cubing this inequality yields the desired result.

Editor's Comment. Two of the other solvers evaluated a, b, c in terms of x, y, z using the Law of Cosines and reduced the problem to showing that the sum of the cosines of the three angles at M was not less than $-3/2$. Michel Bataille noted that this is Problem J341 in *Mathematical Reflections* (3) 2015, whose solution appears in the same journal (4) 2015.

3923. *Proposed by George Apostolopoulos.*

Prove that in any triangle ABC,

$$\frac{\sin^3 \frac{A}{2}}{\sin^3 \frac{A}{2} + \cos^3 \frac{A}{2}} + \frac{\sin^3 \frac{B}{2}}{\sin^3 \frac{B}{2} + \cos^3 \frac{B}{2}} + \frac{\sin^3 \frac{C}{2}}{\sin^3 \frac{C}{2} + \cos^3 \frac{C}{2}} \leq \frac{3R}{2(r + s)},$$

where s, r and R are the semiperimeter, the inradius and the circumradius, respectively, of the triangle ABC.

Editor's Comments: We received three incorrect solutions (two of them were identical) all "proving" the given inequality which actually turned out to be false. One solver attempted to give a counterexample which is not valid. The counterexample below is by Michel Bataille.

To see that the given inequality is incorrect, consider an isosceles right triangle ABC with $\angle A = \frac{\pi}{2}$ and side lengths 1,1 and $\sqrt{2}$. Then using the facts that:

i) $\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$ and $\cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}$,

ii) $R = \dfrac{a}{2\sin\angle A} = \dfrac{\sqrt{2}}{2}$,

iii) $s = \dfrac{2+\sqrt{2}}{2}$,

iv) $r = \dfrac{K}{s} = \dfrac{1}{2+\sqrt{2}} = \dfrac{2-\sqrt{2}}{2}$, where K denotes the area of the triangle,

we compute and find that the left side of the proposed inequality is 0.6326 while the right side is 0.5303 (both rounded to 4 decimal places).

Arslanagić stated, without proof, that the given inequality would hold if the right side is replaced by $\dfrac{2.03R}{r+s}$.

3924. *Proposed by Michel Bataille.*

Let $\{F_k\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for every positive integer k. If m and n are positive integers with m odd and n not a multiple of 3, prove that $5F_m^2 - 3$ divides $5F_{mn}^2 + 3(-1)^n$.

We received three correct solutions and one incorrect solution.

Solution 1, by Oliver Geupel.

Let

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Let $\{L_n\}$ denote the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_n + L_{n+1}$ for every nonnegative integer n. Binet's formulas state that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ for every $n \geq 0$. Let m and k be nonnegative integers with m odd. By Binet's formulas and $\alpha\beta = -1$, we obtain

$$5F_k^2 + 2(-1)^k = 5\left(\frac{\alpha^k + \beta^k}{\alpha - \beta}\right)^2 + 2(-1)^k = \alpha^{2k} + \beta^{2k} = L_{2k}, \tag{1}$$

and furthermore

$$L_{2m}L_{2m(k+1)} = L_{2mk} + L_{2m(k+2)}. \tag{2}$$

Let $P(n)$ denote the assertion that for every odd natural number m,

$$L_{2mn} \equiv \begin{cases} (-1)^{n+1} \pmod{L_{2m} - 1} & \text{if } n \not\equiv 0 \pmod{3}, \\ 2(-1)^n \pmod{L_{2m} - 1} & \text{if } n \equiv 0 \pmod{3}. \end{cases}$$

We will show by induction on n that $P(n)$ is valid for every $n \geq 0$. For the start cases, $L_0 = 2 \equiv 2 \pmod{L_{2m} - 1}$ and $L_{2m} \equiv 1 \pmod{L_{2m} - 1}$, which shows $P(0)$ and $P(1)$.

Given any $n \geq 2$, we show $P(n)$ under the hypotheses $P(n-2)$ and $P(n-1)$ in three cases.

First consider $n \equiv 0 \pmod 3$. By (2) and the induction hypothesis, we obtain

$$L_{2mn} = L_{2m}L_{2m(n-1)} - L_{2m(n-2)} \equiv 1 \cdot (-1)^n - (-1)^{n-1} \quad (3)$$
$$\equiv 2(-1)^n \pmod{L_{2m} - 1}\,. \quad (4)$$

Next let $n \equiv 1 \pmod 3$. Then

$$L_{2mn} \equiv 1 \cdot 2(-1)^{n-1} - (-1)^{n-1} \equiv (-1)^{n+1} \pmod{L_{2m} - 1}\,.$$

Finally consider $n \equiv 2 \pmod 3$. Then

$$L_{2mn} \equiv 1 \cdot (-1)^n - 2(-1)^{n-2} \equiv (-1)^{n+1} \pmod{L_{2m} - 1}\,.$$

Hence $P(n)$ follows and the induction is complete.

Suppose that m and n are positive integers with m odd and n not a multiple of 3. By (1), we obtain $5F_m^2 - 3 = L_{2m} - 1$ and $5F_{mn}^2 + 3(-1)^n = L_{2mn} - (-1)^{n+1}$. By $P(n)$, we conclude that $5F_m^2 - 3$ divides $5F_{mn}^2 + 3(-1)^n$.

Solution 2, by C.R. Pranesachar, slightly expanded by the editor.

For all positive integers m, n with m odd and $n \not\equiv 0 \pmod 3$, let

$$Q_{mn} = \frac{5F_{mn}^2 + 3(-1)^n}{5F_m^2 - 3}.$$

Let $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$, such that $F_t = \frac{x^t - y^t}{\sqrt{5}}$. For any odd integer m,

$$\begin{aligned} 5F_m^2 - 3 &= (x^m - y^m)^2 - 3 = x^{2m} + y^{2m} - 2x^m y^m - 3 \\ &= x^{2m} + y^{2m} - 2(-1)^m - 3 \\ &= x^{2m} + y^{2m} - 1 \\ &= x^{2m} + y^{2m} - x^m y^m. \end{aligned}$$

Furthermore

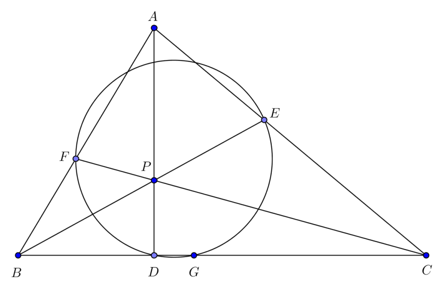
$$\begin{aligned} 5F_{mn}^2 + 3(-1)^n &= (x^{mn} - y^{mn})^2 + 3(-1)^n \\ &= x^{2mn} + y^{2mn} - 2x^{mn}y^{mn} + 3(xy)^{mn} \\ &= x^{2mn} + x^{mn}y^{mn} + y^{2mn}. \end{aligned}$$

We know that for any positive integer n, the polynomial $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ if and only if $n \not\equiv 0 \pmod 3$. This is obtained by a simple application of the Remainder Theorem: if ω is a nonreal cube root of unity, then $(x - \omega)$ and $(x - \omega^2)$ are factors of $x^{2n} + x^n + 1$. Similarly, the polynomial $x^{2n} + x^n y^n + y^2 n$ is divisible by $x^2 + xy + y^2$ for $n \not\equiv 0 \pmod 3$ as it has factors $(x - \omega y)$ and $(x - \omega^2 y)$ as a polynomial in x. Consequently, Q_{mn} is a polynomial in x and y with integer coefficients. Since both its numerator and denominator are symmetric polynomials in x and y, so is Q_{mn}. Hence Q_{mn} can be expressed as a polynomial with integer coefficients in terms of $x + y$ and xy. But $x + y = 1$ and $xy = -1$. Consequently Q_{mn} is an integer, as desired.

As a remark, in the case of $n \equiv 0 \pmod 3$, the term $5F_m^2 - 3$ divides F_{mn}^2.

3925. *Proposed by Ilker Can Çiçek.*

Let ABC be a scalene triangle. Let D be the foot of the altitude from the vertex A. Let P be the point on the segment AD ($P \neq A, P \neq D$), such that for the points E and F defined by $BP \cap AC = E$ and $CP \cap AB = F$, the equality $BF \cdot CD = BD \cdot CE$ holds. Let G be the intersection point of the circumcircle of the triangle DEF and the segment BC with G lying between D and C:



Prove that $AB + AC = BC + AE$ if and only if $BF + CG = CE + BD$.

We received three correct submissions. We present the solution by Peter Y. Woo.

By Ceva's theorem we have

$$\frac{BF}{FA} \cdot \frac{AE}{EC} \cdot \frac{CD}{DB} = 1, \tag{1}$$

so that the given equality $BF \cdot CD = BD \cdot CE$ is equivalent to

$$AE = AF.$$

We shall now see that there is exactly one possibility for the location of P on AD: There is a unique position of E on AC and F on AB satisfying both $AE = AF$ and equation (1) when, as in our problem, the sides $b = AC$ and $c = AB$ have different lengths (and, consequently, $CD \neq DB$). Letting x denote the common length of $AE = AF$ (and assuming, as the proposer intended, that the order of the points on BC is $BDGC$), we see that (1) reduces to

$$x = \frac{b \cdot DB - c \cdot CD}{DB - CD},$$

which is a fixed quantity. Note also that the center of any circle through E and F necessarily lies on the bisector of $\angle BAC$. But we can say more: The angle bisector AG is the diameter of the circumcircle of ΔDEF (see Figure 1).

To see this let Γ be the circle whose diameter is AG. Then Γ contains D (the foot of the altitude from A). It remains to prove that Γ intersects the sides AC

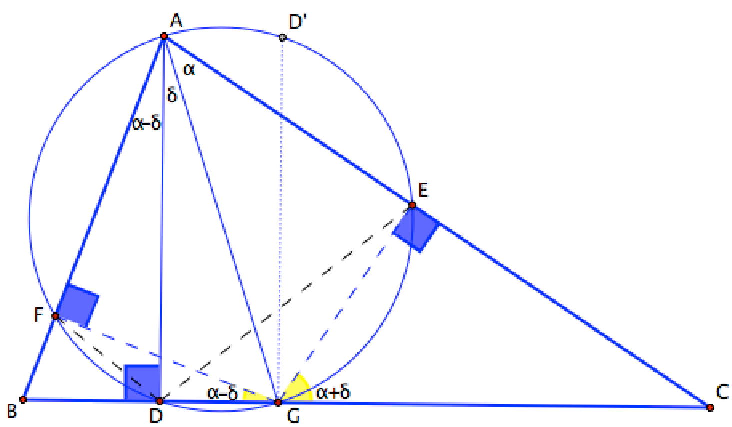


Figure 1: The angle bisector AG is the diameter of the circumcircle of ΔDEF.

and AB at the unique points E and F that satisfy (1) together with $AE = AF$. Then, as in the figure, AFG and AEG are congruent right triangles, and we let $\alpha = \frac{1}{2}\angle BAC$ and $\delta = \angle DFG = \angle DAG = \angle DEG$. Then $\angle BFD = 90^\circ - \delta$ and $\angle BDF = \angle FAG = \alpha$; hence (by the sine law applied to ΔBDF),

$$\frac{BF}{BD} = \frac{\sin\angle BDF}{\sin\angle BFD} = \frac{\sin\alpha}{\cos\delta}.$$

Similarly, $\angle DEC = 90^\circ + \delta$ and $\angle CDE = \angle GDE = \angle GAE = \alpha$, and

$$\frac{CE}{CD} = \frac{\sin\angle CDE}{\sin\angle DEC} = \frac{\sin\alpha}{\cos\delta}.$$

Consequently (1) holds for these positions of E and F, and we proved the claim that these are the points on BP and CP described in the statement of the problem.

As in the figure, we have $\angle FGD = \angle FAD = \alpha - \delta$ and $\angle CGE = \angle DAE = \alpha + \delta$, so that

$$\frac{AB}{AD} = \frac{1}{\cos(\alpha-\delta)}, \quad \frac{AC}{AD} = \frac{1}{\cos(\alpha+\delta)},$$

$$\frac{BC}{AD} = \frac{BD+DC}{AD} = \tan(\alpha-\delta) + \tan(\alpha+\delta),$$

$$\frac{AE}{AD} = \frac{AE}{AG}\cdot\frac{AG}{AD} = \frac{\cos\alpha}{\cos\delta}.$$

The first of the given equations thereby becomes

$$\begin{aligned} 0 &= AB + AC - BC - AE \\ &= AD\left(\frac{1}{\cos(\alpha-\delta)} + \frac{1}{\cos(\alpha+\delta)} - (\tan(\alpha-\delta)+\tan(\alpha+\delta)) - \frac{\cos\alpha}{\cos\delta}\right). \end{aligned}$$

After multiplying and dividing by convenient nonzero terms this becomes, in succession,

$$\begin{aligned}
0 &= \cos(\alpha+\delta)+\cos(\alpha-\delta)-\sin 2\alpha-\frac{\cos\alpha}{\cos\delta}\cdot\frac{1}{2}(\cos 2\alpha+\cos 2\delta)\\
&= 2\cos\alpha\cos\delta-\sin 2\alpha-\frac{\cos\alpha}{2\cos\delta}(\cos 2\alpha+\cos 2\delta),\\
0 &= 4\cos\alpha\cos^2\delta-2\sin 2\alpha\cos\delta-\cos\alpha\cos 2\alpha-\cos\alpha\cos 2\delta,\\
0 &= 4\cos^2\delta-4\sin\alpha\cos\delta-\cos 2\alpha-\cos 2\delta,\\
0 &= 4\cos^2\delta-4\sin\alpha\cos\delta-1+2\sin^2\alpha-2\cos^2\delta+1,\\
0 &= \cos^2\delta-2\sin\alpha\cos\delta+\sin^2\alpha \;=\; (\cos\delta-\sin\alpha)^2.
\end{aligned}$$

By a similar calculation using the ratios

$$\frac{BF}{GF}=\tan(\alpha-\delta),\quad \frac{CE}{GF}=\frac{CE}{GE}=\tan(\alpha+\delta),\quad \frac{CG}{GF}=\frac{CE}{GE}=\frac{1}{\cos(\alpha+\delta)},$$

$$\text{and}\quad \frac{BD}{GF}=\frac{BD}{AD}\cdot\frac{AD}{AG}\cdot\frac{AG}{GF}=\tan(\alpha-\delta)\cdot\cos\delta\cdot\frac{1}{\sin\alpha},$$

the second of the given equations becomes

$$\begin{aligned}
0 &= BF-CE+CG-BD\\
&= GF\left(\tan(\alpha-\delta)-\tan(\alpha+\delta)+\frac{1}{\cos(\alpha+\delta)}-\frac{\cos\delta}{\sin\alpha}\tan(\alpha-\delta)\right),
\end{aligned}$$

whence

$$\begin{aligned}
0 &= -\sin 2\delta+\frac{1}{\sin\alpha}\left(\sin\alpha\cos(\alpha-\delta)-\cos\delta\cos(\alpha+\delta)\sin(\alpha-\delta)\right),\\
0 &= -\sin\alpha\sin 2\delta+\sin\alpha(\cos\alpha\cos\delta+\sin\alpha\sin\delta)-\cos\delta\cdot\frac{1}{2}\cdot(\sin 2\alpha-\sin 2\delta)\\
&= -\sin\alpha\sin 2\delta+\sin\alpha\cos\alpha\cos\delta+\sin^2\alpha\sin\delta-\cos\delta\sin\alpha\cos\alpha+\frac{\cos\delta\sin 2\delta}{2},\\
0 &= -\sin\alpha+\frac{\sin^2\alpha}{2\cos\delta}+\frac{\cos\delta}{2},\\
0 &= -2\sin\alpha\cos\delta+\sin^2\alpha+\cos^2\delta \;=\; (\cos\delta-\sin\alpha)^2.
\end{aligned}$$

We conclude, finally, that $AB+AC=BC+AE$ and $BF+CG=CE+BD$ are both equivalent to $\cos\delta=\sin\alpha$.

Editor's Comments. It seems that after all this work, we have managed to discover yet another fascinating property of the empty set: The required condition (namely, $\cos\delta=\sin\alpha$) *can never* occur (unless, of course, this editor has made an error, which *can* occur — frequently). Note that in ΔADG we have $\cos\delta=\frac{AD}{AG}$, while in ΔAGE we have $\sin\alpha=\frac{GE}{AG}$. But equality would imply that $AD=GE$. Letting D' be the point of the circle Γ diametrically opposed to D, we have $AD=GD'>GE$ (because D' is outside the circle with centre G that meets Γ in E and F). In other words, there exists no triangle that satisfies all the requirements of the problem. Specifically, our assumptions (namely $AC > AB, BF \cdot CD = BD \cdot CE$, and the points along BC lie in the order $BDGC$) imply that $AB + AC > BC + AE$ and $BF + CG > CE + BD$.

3926. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers such that $a+b+c = 1$. Find the minimum value of the expression

$$\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}.$$

We received twelve correct submissions. We present 3 different solutions.

Solution 1, by Michel Bataille.

For $x, y > 0$, we have $2(x^2 + y^2) \geq (x + y)^2$, hence

$$x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}=(x^2+y^2)\left(1+\frac{1}{x^2y^2}\right)\geq\frac{(x+y)^2}{2}\left(1+\frac{1}{(xy)^2}\right).$$

It follows that the given expression E satisfies $E \geq E'$ where

$$E'=\sqrt{2}\left(\frac{a+b}{2}\sqrt{1+\frac{1}{a^2b^2}}+\frac{b+c}{2}\sqrt{1+\frac{1}{b^2c^2}}+\frac{c+a}{2}\sqrt{1+\frac{1}{c^2a^2}}\right).$$

Consider the function $f(t) = \sqrt{1 + t^2}, t > 0$. Then straightforward calculations show that

$$f'(t)=\frac{t}{\sqrt{1+t^2}}\quad\text{and}\quad f''(t)=\frac{1}{(1+t^2)^{3/2}}>0,$$

so f is convex on $(0, \infty)$.

Since

$$\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}=1,$$

Jensen's inequality yields

$$E'\geq\sqrt{2}\cdot\sqrt{1+\left(\frac{a+b}{2ab}+\frac{b+c}{2bc}+\frac{c+a}{2ca}\right)^2}=\sqrt{2}\cdot\sqrt{1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}.$$

Since

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{9}{a+b+c}=9$$

by the AM-HM inequality, we finally obtain $E \geq E' \geq \sqrt{2}\sqrt{82} \geq 2\sqrt{41}$.

It is easily checked that equality holds if and only if $a = b = c = \frac{1}{3}$.

Solution 2, by John G. Heuver.

By Minkowski's inequality and the AM-GM inequality, we have:

$$\begin{aligned}
&\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}\\
&\qquad\qquad \geq \sqrt{(a+b+c)^2+(b+c+a)^2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\\
&\qquad\qquad = \sqrt{2+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2} \geq \sqrt{2+2\left(\frac{3}{\sqrt[3]{abc}}\right)^2}\\
&\qquad\qquad \geq \sqrt{2+2\left(\frac{9}{a+b+c}\right)^2} = \sqrt{2+2\cdot 81} = \sqrt{164} = 2\sqrt{41}.
\end{aligned}$$

Solution 3, by Titu Zvonaru.

Since $2(x^2+y^2) \geq (x+y)^2$, we have $\sqrt{x^2+y^2} \geq \frac{1}{\sqrt{2}}(x+y)$. By the AM-HM inequality, we also have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c}.$$

Hence by using the triangle inequality for complex numbers, we obtain:

$$\begin{aligned}
&\sqrt{a^2+b^2+\frac{1}{a^2}+\frac{1}{b^2}}+\sqrt{b^2+c^2+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{a^2+c^2+\frac{1}{a^2}+\frac{1}{c^2}}\\
&\quad = \sqrt{\left|a+\frac{i}{a}\right|^2+\left|b+\frac{i}{b}\right|^2}+\sqrt{\left|b+\frac{i}{b}\right|^2+\left|c+\frac{i}{c}\right|^2}+\sqrt{\left|a+\frac{i}{a}\right|^2+\left|c+\frac{i}{c}\right|^2}\\
&\quad \geq \frac{1}{\sqrt{2}}\left(2\left|a+\frac{i}{a}\right|+2\left|b+\frac{i}{b}\right|+\left|c+\frac{i}{c}\right|\right)\\
&\quad \geq \frac{2}{\sqrt{2}}\left|(a+b+c)+i\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right| = \sqrt{2}\sqrt{(a+b+c)^2+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\\
&\quad \geq \sqrt{2}\sqrt{1+\left(\frac{9}{a+b+c}\right)^2} = \sqrt{2}\sqrt{82} = 2\sqrt{41}.
\end{aligned}$$

3927. *Proposed by Marcel Chiriţă.*

Let $ABCO$ be a tetrahedron with the face angles at O all right angles. If we denote the altitude from O by h, the inradius by r, and the angles that the lines OA, OB, OC make with the face ABC by x, y, z, show that

$$r\sqrt{2} \geq h(\cos 2x + \cos 2y + \cos 2z).$$

We received three correct submissions. We present the solution by Titu Zvonaru.

The proposed inequality is false; we shall see that instead

$$r(1+\sqrt{3}) \geq h(\cos 2x + \cos 2y + \cos 2z).$$

We use square brackets to denote areas and denote by V the volume of $ABCO$; letting $a = OA, b = OB$, and $c = OC$, we have

$$V = \frac{[OAB]\cdot OC}{3} = \frac{[OBC]\cdot OA}{3} = \frac{[OCA]\cdot OB}{3} = \frac{abc}{6}.$$

Because $AB^2 = a^2+b^2,\ BC^2 = b^2+c^2,\ CA^2 = c^2+a^2$, we have

$$\begin{aligned} 16[ABC]^2 &= 2(a^2+b^2)(b^2+c^2)+2(b^2+c^2)(c^2+a^2)+2(c^2+a^2)(a^2+b^2) \\ &\qquad -(a^2+b^2)^2-(b^2+c^2)^2-(c^2+a^2)^2 \\ &= 4(a^2b^2+b^2c^2+c^2a^2). \end{aligned}$$

In terms of h and r the volume is

$$V = \frac{[ABC]\cdot h}{3} = \frac{r([OAB]+[OBC]+[OCA]+[ABC])}{3};$$

consequently,

$$h = \frac{abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}} \quad \text{and} \quad r = \frac{abc}{ab+bc+ca+\sqrt{a^2b^2+b^2c^2+c^2a^2}}.$$

Let H be the foot of the altitude from O to the face ABC. Then $\sin x = \frac{OH}{OA} = \frac{h}{a}$, and

$$\sin^2 x = \frac{h^2}{a^2} = \frac{b^2c^2}{a^2b^2+b^2c^2+c^2a^2},$$

so that

$$\cos 2x = 1 - 2\sin^2 x = 1 - \frac{2b^2c^2}{a^2b^2+b^2c^2+c^2a^2}.$$

Add this last equation to the analogous expressions for $\cos 2y$ and $\cos 2z$ to get

$$\cos 2x + \cos 2y + \cos 2z = 1.$$

We can now prove our initial claim:

$$\begin{aligned} & r(1+\sqrt{3}) \geq h(\cos 2x+\cos 2y+\cos 2z) \\ \Leftrightarrow\quad & \frac{1+\sqrt{3}}{ab+bc+ca+\sqrt{a^2b^2+b^2c^2+c^2a^2}} \geq \frac{1}{\sqrt{a^2b^2+b^2c^2+c^2a^2}} \\ \Leftrightarrow\quad & \sqrt{3(a^2b^2+b^2c^2+c^2a^2)} \geq ab+bc+ca \\ \Leftrightarrow\quad & 3(a^2b^2+b^2c^2+c^2a^2) \geq (ab+bc+ca)^2 \\ \Leftrightarrow\quad & (ab-bc)^2+(bc-ca)^2+(ca-ab)^2 \geq 0. \end{aligned}$$

Equality holds if and only if $a = b = c$.

Editor's Comments. The editors misinterpreted Chiriţă's original proposal: he intended r to be the radius of the incircle of the face ΔABC (not the inradius of the tetrahedron). Using an argument similar to that of the featured solution with his inradius, he correctly obtained the inequality of his proposal, namely $r\sqrt{2} \geq h(\cos 2x + \cos 2y + \cos 2z)$, with equality if and only if $a = b = c$.

3928. *Proposed by Michel Bataille.*

Let $A \in \mathcal{M}_n(\mathbb{C})$ with $\operatorname{rank}(A) \leq 1$ and complex numbers $x_1, x_2, \ldots, x_n$ such that $\sum_{k=1}^{n} x_k^2 = 1$. If

$$B = \left(\begin{array}{c|c} 0 & x_1 \cdots x_n \\ \hline x_1 & \\ \vdots & A \\ x_n & \end{array}\right)$$

and I_{n+1} is the unit matrix of size $n+1$, prove that

$$\det(I_{n+1} + B) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

We received three correct solutions. We present the solution submitted by Oliver Geupel, slightly modified by the editor.

Since $\operatorname{rank}(A) \leq 1$, there are complex numbers $a_1, \ldots, a_n$ and $\lambda_1, \ldots, \lambda_n$ such that

$$A = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} = \begin{pmatrix} \lambda_1 a_1 & \cdots & \lambda_1 a_n \\ \vdots & & \vdots \\ \lambda_n a_1 & \cdots & \lambda_n a_n \end{pmatrix}.$$

Denote by I_n the unit matrix of size n. We construct two intermediate matrices C and D, and relate the determinants of C, D and B to each other by showing how each matrix can be built up from the previous via elementary row operations. Let

$$C = \left(\begin{array}{c|c} 1 & a_1 \cdots a_n \\ \hline \lambda_1 & \\ \vdots & I_n + A \\ \lambda_n & \end{array}\right) = \begin{pmatrix} C_0 & | & \cdots & | & C_n \end{pmatrix}$$

(that is, $C_0, \ldots, C_n$ denote the columns of C); we obtain

$$\begin{aligned} \det C &= \det \begin{pmatrix} C_0 & | & C_1 - a_1 C_0 & | & \cdots & | & C_n - a_n C_0 \end{pmatrix} \\ &= \det \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \lambda_1 & \\ \vdots & I_n \\ \lambda_n & \end{array}\right) = 1. \end{aligned}$$

Let

$$D = \left(\begin{array}{c|c} 0 & a_1 \cdots a_n \\ \hline x_1 & \\ \vdots & I_n + A \\ x_n & \end{array}\right) = \left(\begin{array}{c|c|c} D_0 & \cdots & D_n \end{array}\right)$$

(that is, $D_0, \ldots, D_n$ denote the columns of D). Note that $D_i = C_i$ for $i \geq 1$. On the other hand, a simple calculation shows that

$$\begin{aligned} & D_0 - x_1 D_1 - \cdots - x_n D_n \\ & \quad = \begin{pmatrix} -(a_1x_1 + \cdots + a_nx_n) \\ -\lambda_1(a_1x_1 + \cdots + a_nx_n) \\ \vdots \\ -\lambda_n(a_1x_1 + \cdots + a_nx_n) \end{pmatrix} = -(a_1x_1 + \cdots + a_nx_n)C_0. \end{aligned}$$

Using the properties of determinants,

$$\begin{aligned} \det D &= \det \left(\begin{array}{c|c|c|c} D_0 - x_1D_1 - \cdots - x_nD_n & D_1 & \cdots & D_n \end{array}\right) \\ &= \det \left(\begin{array}{c|c|c|c} -(a_1x_1 + \cdots + a_nx_n)C_0 & C_1 & \cdots & C_n \end{array}\right) \\ &= -(a_1x_1 + \cdots + a_nx_n) \det C \\ &= -(a_1x_1 + \cdots + a_nx_n). \end{aligned}$$

Let

$$I_{n+1} + B = \left(\begin{array}{c} R_0 \\ \hline \vdots \\ \hline R_n \end{array}\right)$$

so $R_0, \ldots, R_n$ denote the rows of $I_{n+1} + B$. Note that, except for the first row, the rows of $I_{n+1} + B$ are the same as the rows of D. Moreover,

$$\begin{aligned} & R_0 - x_1R_1 - \cdots - x_nR_n \\ & \quad = \left(0 \quad -a_1(\lambda_1x_1 + \cdots + \lambda_nx_n) \cdots - a_n(\lambda_1x_1 + \cdots + \lambda_nx_n) \right), \end{aligned}$$

which is $-(\lambda_1x_1 + \cdots + \lambda_nx_n)$ times the first row of D. Note that for the above calculation we needed to use the hypothesis that $\sum\limits_{k=1}^{n} x_k^2 = 1$ to get that the first entry is zero. By the properties of determinants, it follows that

$$\begin{aligned}
\det(I_{n+1}+B) &= \det\begin{pmatrix} R_0 - x_1R_1 - \cdots - x_nR_n \\ \hline R_1 \\ \hline \vdots \\ \hline R_n \end{pmatrix} \\
&= -\left(\lambda_1 x_1 + \cdots + \lambda_n x_n\right)\det D \\
&= (\lambda_1 x_1 + \cdots + \lambda_n x_n)(a_1x_1 + \cdots + a_nx_n) \\
&= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
&= \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.
\end{aligned}$$

This completes the proof.

3929. *Proposed by Péter Ivády.*

Show that for all $0 < x < \pi/2$, the following inequality holds:

$$\left(1+\frac{1}{\sin x}\right)\left(1+\frac{1}{\cos x}\right) \geq 5\left[1+x^4\left(\frac{\pi}{2}-x\right)^4\right].$$

There were eight submissions for this problem, all of which were correct. We present the solution by C.R. Pranesachar.

We shall prove that if

$$f(x) = \left(1+\frac{1}{\sin(x)}\right)\left(1+\frac{1}{\cos(x)}\right), \text{ and } g(x) = 5 + x^5\left(\frac{\pi}{4}-x\right)^4, \ 0 < x < \frac{\pi}{2},$$

then

$$\min_{0<x<\frac{\pi}{2}} f(x) > 5.8 > \max_{0<x<\frac{\pi}{2}} g(x).$$

We shall give a calculus-free proof. Since $f(x)$ is symmetric about the point $x = \frac{\pi}{4}$

in $(0, \frac{\pi}{2})$, we may use the substitution $x = \frac{\pi}{4} - t$, where $-\frac{\pi}{4} < t < \frac{\pi}{4}$. Then

$$\begin{aligned} f(x) &= \left(1 + \frac{1}{\sin(\frac{\pi}{4} - t)}\right)\left(1 + \frac{1}{\cos(\frac{\pi}{4} - t)}\right) \\ &= \frac{\left(\frac{1}{\sqrt{2}}(\cos(t) - \sin(t)) + 1\right)\left(\frac{1}{\sqrt{2}}(\cos(t) + \sin(t)) + 1\right)}{\sin(\frac{\pi}{4} - t)\cos(\frac{\pi}{4} - t)} \\ &= \frac{(\sqrt{2} + \cos(t) - \sin(t))(\sqrt{2} + \cos(t) + \sin(t))}{2\sin(\frac{\pi}{4} - t)\cos(\frac{\pi}{4} - t)} \\ &= \frac{(\sqrt{2} + \cos(t))^2 - \sin^2(t)}{\sin(\frac{\pi}{2} - t)} = \frac{2 + 2\sqrt{2}\cos(t) + \cos(2t)}{\cos(2t)} \\ &= 1 + \frac{2(\sqrt{2}\cos(t) + 1)}{2\cos^2(t) - 1} = 1 + \frac{2}{\sqrt{2}\cos(t) - 1}. \end{aligned}$$

For $f(x)$ to be at a minimum, $\sqrt{2}\cos(t) - 1$ is at a maximum, and so $\cos(t) = 1$. This happens for $t = 0$, that is, $x = \frac{\pi}{4}$. Thus

$$\min f(x) = 1 + \frac{2}{\sqrt{2} - 1} = 3 + 2\sqrt{2} > 3 + 2(1.4) = 5.8.$$

Now, the maximum of $x(\frac{\pi}{2} - x)$ is $\frac{\pi^2}{16}$, which is attained at $x = \frac{\pi}{4}$, as

$$x\left(\frac{\pi}{2} - x\right) = \frac{\pi^2}{16} - \left(\frac{\pi}{4} - x\right)^2.$$

So

$$\max g(x) = 5 + \left(\frac{\pi^2}{16}\right)^4 = 5 + \frac{\pi^8}{16^4}.$$

Since $\pi^2 < 10$ (this follows from the fact that $\pi < 3.15$), we see that:

$$\begin{aligned} \max g(x) < 5\left(1 + \frac{10^4}{16^4}\right) &= 5\left(1 + \frac{10^6}{16^4 \times 100}\right) \\ &= 5\left(1 + \frac{(10^3)^2}{2^16 \times 100}\right) \\ &< 5\left(1 + \frac{(2^10)^2}{2^16 \times 100}\right) \\ &= 5\left(1 + \frac{2^4}{100}\right) = 5(1 + 0.16) = 5.8. \end{aligned}$$

Hence the inequality follows.

3930. *Proposed by José Luis Díaz–Barrero.*

In a triangle ABC, let a, b and c denote the lengths of the sides BC, CA and AB. Show that

$$\sqrt{\frac{a\,\sin^{1/2}B}{4a + b + c}} + \sqrt{\frac{b\,\sin^{1/2}C}{a + 4b + c}} + \sqrt{\frac{c\,\sin^{1/2}A}{a + b + 4c}} \le \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$

We received nine correct submissions. We present the solution by Cao Minh Quang.

Let
$$F = \sum_{\text{cyc}} \sqrt{\frac{a\sin^{1/2} B}{4a+b+c}}.$$

Then by Cauchy-Schwarz inequality, we have
$$F^2 \le \sum_{\text{cyc}} \frac{a}{4a+b+c} \cdot \sum_{\text{cyc}} \sin^{1/2} A. \tag{1}$$

Since $(x+y)^2 \ge 4xy$, we have $\frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$ for all $x>0, y>0$. Hence,
$$\begin{aligned}\sum_{\text{cyc}} \frac{a}{4a+b+c} &= \sum_{\text{cyc}} \frac{a}{3a+(a+b+c)} \\ &\le \frac{1}{4}\sum_{\text{cyc}} a\left(\frac{1}{3a}+\frac{1}{a+b+c}\right) = \frac{1}{4}\sum_{\text{cyc}}\left(\frac{1}{3}+\frac{a}{a+b+c}\right) = \frac{1}{2}. \end{aligned} \tag{2}$$

Since it is well known that $\sum_{\text{cyc}} \sin A \le \frac{3\sqrt{3}}{2}$ we have, by using Cauchy-Schwarz inequality again, that
$$\sum_{\text{cyc}} \sin^{1/2} A \le \left(\sum_{\text{cyc}} 1\right)^{1/2} \left(\sum_{\text{cyc}} \sin A\right)^{1/2} \le \sqrt{3}\sqrt{\frac{3\sqrt{3}}{2}} = \sqrt{\frac{9\sqrt{3}}{2}}. \tag{3}$$

From (1)–(3), we obtain that
$$F^2 \le \frac{1}{2}\sqrt{\frac{9\sqrt{3}}{2}} = \sqrt{\frac{9\sqrt{3}}{8}},$$

therefore,
$$F \le \sqrt[4]{\frac{9\sqrt{3}}{8}} = \sqrt[8]{\frac{3^5}{2^6}} = \frac{3}{2}\sqrt[8]{\frac{4}{27}}.$$

AUTHORS' INDEX

Solvers and proposers appearing in this issue

(Bold font indicates featured solution.)

Proposers

Solvers - individuals

Solvers - collaborations