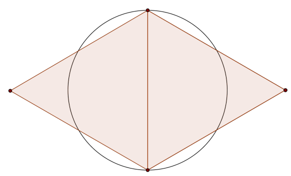
CC163. Si on choisit x aléatoirement dans $[-100, 100]$, quelle est la probabilité que $g[f(x)]$ est négatif compte tenu que $f(x) = x^2 + 3x - 7$ et $g(x) = x^2 - 2x - 99$?

CC164. Sur le diamètre d'un cercle de rayon 5, on construit 2 triangles équilatéraux. Quelle est l'aire totale du cercle en dehors des triangles équilatéraux?



CC165. Georges paye 50$ à chacune de quatre visites à une station-service lorsque les prix par litre étaient 1,32 $, 1,25 $, 1,11 $ et 1,18 $ pendant une période de grande fluctuation de prix. Quel était le prix moyen par litre?

CONTEST CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2014: 40(3), p. 96–97.

CC111. Find all positive integers with two or more digits such that if we insert a 0 between the units and tens digits we get a multiple of the original number.

Originally problem A1 from the 2003 Mexican Math Olympiad.

We received ten correct submissions. We present the solution by Yihang Dong.

Call the number $\overline{xy}$, where y is the one's digit and x is the rest. Then we are looking for numbers such that $k(10x+y) = 100x+y$. If $k > 10$, then $10kx > 100x$ and $ky > y$, which cannot happen. So $k \leq 10$.

Considering mod 10, we see that $ky \equiv y \pmod{10}$. So $10a = ky - y$ for some a. One possibility is $y = 0$. If $y \neq 0$, then $(k-1)y = 10a$. So $10 \mid (k-1)y$. If $5 \mid y$, then $y = 5$. If $5 \mid (k-1)$, then $k - 1 = 5$ and so $k = 6$.

Let us go through these three possibilities.

1. If $y = 0$, then $10kx = 100x$ has solution $k = 10$ for all integers x. So any multiple of 10 works.

2. If $y = 5$, then $k(10x + 5) = 100x + 5$. And so $x = \frac{k-1}{20-2k}$, which means that $k - 1 \geq 20 - 2k$, giving $k \geq 7$. We also have that $2 \mid (k - 1)$, so $k = 7$ or $k = 9$. These lead to $x = 1$ or $x = 4$, leading to solutions 15 and 45.

3. If $k = 6$, then $60x + 6y = 100x + y$, or rather $y = 8x$. Thus $x = 1, y = 8$ and we get the solution 18.

In conclusion, the possibilities are $15, 18, 45$ and multiples of 10.

CC112. Jerome groups odd numbers in groups that contain successive quantities of odd number of elements such as:

$$\{1\}, \{3, 5\}, \{7, 9, 11\}, \{13, 15, 17, 19\}, \ldots$$

What is the sum of the 100th grouping?

Inspired by problem 13 from the second round of South African school Olympiads 2014.

We received twelve correct solutions and two incorrect submissions. We present the solution by Matei Coiculescu.

Because the number of elements in each group increases by 1, for every group, the number of elements until the 100th grouping is

$$1 + 2 + 3 + \cdots + 99 = \frac{99 \cdot 100}{2} = 4950.$$

This means that the sum of the elements of the 100th grouping is the sum of the odd numbers from the 4951st odd number to the 5050th odd number. This sum can be expressed as

$$\sum_{k=4951}^{5050} (2k - 1) = \sum_{k=1}^{5050} (2k - 1) - \sum_{k=1}^{4950} (2k - 1) = 5050^2 - 4950^2 = 1000000.$$

CC113. If P is a point inside $ABCD$ with $PA = 2$, $PB = 3$, $PC = 5$ and $PD = 6$, what is the maximum possible area of $ABCD$?

Originally problem 20 from the qualifying round of South African school Olympiads 2012.

We received five correct solutions, and one incomplete submission. We present two solutions: one starting from the sine formula for area (a feature of all but one of the correct solutions), and one using cross products.

Solution 1, by Hessami Pilehroon Elnaz.

The area of quadrilateral $ABCD$ is equal to

$$\begin{aligned}[APB] + [BPC] + [CPD] + [DPA] & \\ &= \frac{AP \cdot PB \cdot \sin\angle APB}{2} + \frac{BP \cdot PC \cdot \sin\angle BPC}{2} \\ &+ \frac{CP \cdot PD \cdot \sin\angle CPD}{2} + \frac{DP \cdot AP \cdot \sin\angle DPA}{2}\end{aligned}$$

Substituting $AP = 2$, $BP = 3$, $CP = 5$ and $DP = 6$, we get

$$[ABCD] = 3\sin\angle APB + 7.5\sin\angle BPC + 15\sin\angle DPC + 6\sin\angle APD.$$

The maximum value of $\sin\alpha$ is 1 and occurs when $\alpha = 90^\circ$. Therefore, the maximum value of $[ABCD]$ is $3+7.5+15+6 = 31.5$, and it occurs when $\angle APB = \angle BPC = \angle DPC = \angle APD = 90^\circ$. The four 90° angles make 360° at P, so such a quadrilateral can be constructed as follows: draw two perpendicular lines intersecting at P, and measure the correct distances from P to determine the vertices A, B, C and D. Hence the maximum area of $ABCD$ is 31.5.

Solution 2, by Somasundaram Muralidhanan.

Let P be the origin of vectors and let $\overrightarrow{PA} = \mathbf{a}$, $\overrightarrow{PB} = \mathbf{b}$, $\overrightarrow{PC} = \mathbf{c}$ and $\overrightarrow{PD} = \mathbf{d}$. The area of triangle APB, for example, is then $\frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$, where $\times$ denotes the cross product and $\|\cdot\|$ the vector norm. Adding up the area of the triangles which make up $ABCD$, we get

$$S = \frac{1}{2}\left(\|\mathbf{a}\times\mathbf{b}\| + \|\mathbf{b}\times\mathbf{c}\| + \|\mathbf{c}\times\mathbf{d}\| + \|\mathbf{d}\times\mathbf{a}\|\right).$$

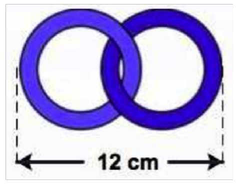
Note that all the cross product vectors in the above expression point in the same direction, so we can combine the norms; that is,

$$\begin{aligned}S &= \frac{1}{2}\|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{d} + \mathbf{d}\times\mathbf{a}\| \\ &= \frac{1}{2}\|\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{d} + \mathbf{c}\times\mathbf{d} - \mathbf{c}\times\mathbf{b}\| \\ &= \frac{1}{2}\|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{d})\| \\ &= \frac{1}{2}\|\mathbf{a}-\mathbf{c}\|\cdot\|\mathbf{b}-\mathbf{d}\|\cdot\sin\theta\end{aligned}$$

where θ is the angle between AC and BD. Clearly, $\|\mathbf{a}-\mathbf{c}\|$ is maximum when P, A, C are collinear and $\|\mathbf{b}-\mathbf{d}\|$ is maximum when P, B, D are collinear. In addition, since the maximum value for $\sin\theta$ is 1, it follows that S is maximum when AC is perpendicular to BD. Thus the maximum area is

$$\frac{1}{2}(2\cdot 3 + 3\cdot 5 + 5\cdot 6 + 6\cdot 2) = \frac{63}{2}.$$

CC114. A chain with two links is 12 cm long. A chain with five links is 27 cm long. What is the length, in cm, of a chain with 40 links?



Originally problem 19 from 2012 South African school Olympiads.

We received nine correct solutions. We present the solution by Yihang Dong.

Let a be the diameter of a link, and let b be the overlap of 2 links. Then we have the system of equations

$$2a - b = 12$$
$$5a - 4b = 27,$$

which has solution $a = 7$ and $b = 2$. A chain with 40 links will have 39 overlaps, leading to $40a - 39b = 202$ cm. Therefore a 40 link chain will be 202 cm long.

CC115. Mathias has put together 120 identical unit cubes to form a rectangular prism and painted all six sides of it. There are 24 unpainted cubes left when the prism is undone. What is the surface area of the prism?

Originally problem 20 from 2015 South African school Olympiads.

We received four correct solutions and one incorrect submission. We present the solution by David Lowry-Duda.

We think of the prism as an inner core of 24 cubes in a prism surrounded by a shell of the remaining cubes in a larger prism.

Then if the dimensions of the inner core are $lwh = 24$, the dimensions of the entire prism are $(l+2)(w+2)(h+2) = 120$. Note that $120 = 2^3 \cdot 3 \cdot 5$ and $24 = 2^3 \cdot 3$. Then we must choose l, w, h from the factors of 24 so that both $lwh = 24$ and $(l+2)(w+2)(h+2) = 120$ with extended side lengths appearing as factors of 120.

Since 5 must divide one of $l+2, w+2, h+2$, we can conclude without loss of generality that $l = 3$, as this is the only way to yield exactly one multiple of 5 (the other case produces no integer solutions). Then $wh = 8$ and $(w+2)(h+2) = wh + 2w + 2h + 4 = 24$, or rather $2w + 2h = 12$. Since the possibilities for h, w are either $1, 2^3$ or $2, 2^2$ and $2w + 2h = 12$, we must have (without loss of generality) that $w = 4, h = 2$.

So the inner core is a $2 \times 3 \times 4$ prism, and the full prism is $4 \times 5 \times 6$. The surface area is $2 \cdot (20) + 2 \cdot (30) + 2 \cdot (24) = 148$.