OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2014 : 40(8), p. 326–327.

OC196. The number 5654 in base b is a power of a prime number. Find b if $b > 6$.

Originally problem 4 from the 2013 Italian Mathematical Olympiad.

We received five correct submissions. We present the solution by the Missouri State University Problem Solving Group.

We must find b such that

$$5b^3 + 6b^2 + 5n + 4 = (b + 1)(5b^2 + b + 4) = p^n$$

where p is prime. By unique factorization, we must have that $b + 1 = p^\alpha$ and $5b^2 + b + 4 = p^\beta$, where $\alpha + \beta = n$. Solving for b in the first equation and substituting in the second gives

$$5(p^\alpha - 1)^2 + (p^\alpha - 1) + 4 = p^\beta$$

Now, $5b^2 + b + 4 > b + 1$, so $\beta > \alpha$. Reducing our equation modulo p^α yields $5 - 1 + 4 = 8 \equiv 0 \mod p^\alpha$. This forces $p^\alpha \mid 8$ and since $b > 6$, our only possibility is $p^\alpha = 8$ giving $b = 7$. In this case,

$$5 \cdot 7^3 + 6 \cdot 7^2 + 5 \cdot 7 + 4 = 2048 = 2^{11}$$

and hence $b = 7$ is the only solution.

OC197. A $n \times n \times n$ cube is constructed using $1 \times 1 \times 1$ cubes, some of them black and others white, such that in each $n \times 1 \times 1$, $1 \times n \times 1$, and $1 \times 1 \times n$ subprism there are exactly two black cubes, and they are separated by an even number of white cubes (possibly 0). Show it is possible to replace half of the black cubes with white cubes such that each $n \times 1 \times 1$, $1 \times n \times 1$ and $1 \times 1 \times n$ subprism contains exactly one black cube.

Originally problem 4 from the 2013 Mexico National Olympiad.

We received no solutions to this problem.

OC198. Determine all positive real M such that for any positive reals a, b, c, at least one of

$$a + \frac{M}{ab}, b + \frac{M}{bc}, c + \frac{M}{ca}$$

is greater than or equal to $1 + M$.

Originally problem 3 from day 1 of the 2013 Indonesia Mathematical Olympiad.

We received three correct submissions. We present the solution by Titu Zvonaru.

Let $0 < \epsilon < 1$ be an arbitrary real number. Let $a = b = c = 1 + \epsilon$. We obtain

$$1 + \epsilon + \frac{M}{(1+\epsilon)^2} \geq 1 + M$$
$$\epsilon \geq \frac{M(\epsilon^2 + 2\epsilon)}{(1+\epsilon)^2}$$
$$\frac{(1+\epsilon)^2}{2+\epsilon} \geq M$$

Similarly, for $a = b = c = 1 - \epsilon$, it follows that

$$\frac{(1-\epsilon)^2}{2-\epsilon} \leq M.$$

As ϵ tends to 0, the above two inequalities imply that we must necessarily have $M = 1/2$ as the only valid solution.

In order to prove that $M = 1/2$ satisfies the statement of the problem, it suffices to show that

$$a + b + c + \frac{1}{2}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 3\left(1 + \frac{1}{2}\right)$$

which is equivalent to showing that

$$a + b + c \geq \frac{9abc}{2abc + 1}.$$

By the AM-GM inequality, we have $a + b + c \geq 3\sqrt[3]{abc}$. Thus, it suffices to show that

$$3\sqrt[3]{abc} \geq \frac{9abc}{2abc + 1}.$$

Isolating shows that

$$2(abc) - 3(abc)^{2/3} + 1 \geq 0$$

Factoring yields that the above is equivalent to

$$(\sqrt[3]{abc} - 1)^2(2\sqrt[3]{abc} + 1) \geq 0$$

which is true for any positive real values of a, b and c.

OC199. Determine all pairs of polynomials f and g with real coefficients such that

$$x^2 \cdot g(x) = f(g(x)).$$

Originally problem 4 from the 2013 South Africa National Olympiad.

We received four correct submissions and two incorrect submissions. We present the solution by Michel Bataille.

We show that the solutions are of three types :

(1) g is the zero polynomial and f is any polynomial divisible by polynomial x.

(2) $g(x) = ax + b$ and $f(x) = \frac{x(x-b)^2}{a^2}$ for some real numbers a, b with $a \neq 0$.

(3) $g(x) = ax^2 + b$ and $f(x) = \frac{x(x-b)}{a}$ for some real numbers a, b with $a \neq 0$.

It is readily checked that in cases $(1), (2), (3)$, the pair (f, g) is a solution.

Conversely, let (f, g) be a solution. If $g(x)$ is the zero polynomial, then $f(0) = 0$, hence f is divisible by x. Otherwise, f is not the zero polynomial either and we will denote by n and k the degrees of f and g, respectively. Since the respective degrees of $x^2 \cdot g(x)$ and $f(g(x))$ are $k + 2$ and kn, we must have $k + 2 = kn$, that is, $(n - 1)k = 2$. It follows that either $k = 1,\ n = 3$ or $k = n = 2$.

In the former case, let $g(x) = ax + b$ where $a \neq 0$. Then,

$$x^2 g(x) = x^2(ax + b) = f(ax + b) = f(b) + axf_1(b) + q(x)$$

where $q(x)$ is a polynomial divisible by x^2 and $f_1(x)$ is some polynomial. It follows that $f(b) = f_1(b) = 0$. Thus, $f(x)$ is divisible by $(x - b)^2$. Setting

$$f(x) = (x - b)^2(ux + v) \quad \text{and} \quad x^2 g(x) = f(ax + b)$$

now gives $ua^3 = a$ and $(bu + v)a^2 = b$ so that $u = \frac{1}{a^2}$ and $v = 0$ and we conclude that $f(x) = \frac{x(x-b)^2}{a^2}$.

In the latter case, substituting a root (possibly complex) of $g(x)$ for x in the equality $x^2 g(x) = f(g(x))$ we obtain $f(0) = 0$. As a result, $f(x)$ is divisible by x. Setting

$$f(x) = \alpha x^2 + \beta x \quad \text{and} \quad g(x) = ax^2 + bx + c$$

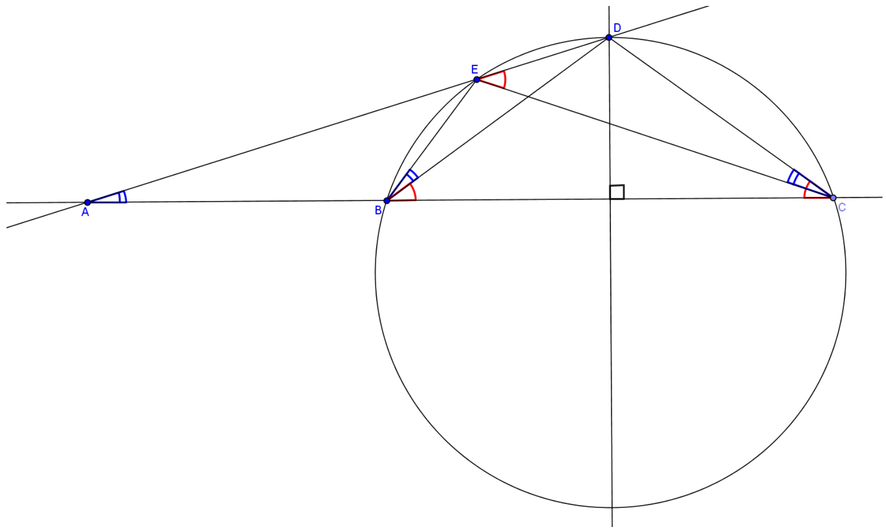
(with $a, \alpha \neq 0$), the equality $x^2 g(x) = f(g(x))$ yields $x^2 = \alpha(ax^2 + bx + c) + \beta$ and we deduce $\alpha = \frac{1}{a},\ b = 0,\ \beta = -\frac{c}{a}$. Therefore $f(x) = \frac{x(x-c)}{a}$ and $g(x) = ax^2 + c$, a pair of type (3).

Editor's Comments. We received four solutions neglecting the zero polynomial case. They were considered correct though only the solution given below included the case when $g(x) = 0$.

OC200. Let A, B and C be three points on a line (in this order). For each circle k through the points B and C, let D be one point of intersection of the perpendicular bisector of BC with the circle k. Further, let E be the second point of intersection of the line AD with k. Show that for each circle k, the ratio of lengths $BE : CE$ is the same.

Originally problem 4 from part 1 of the 2013 Austrian Federal Competition For Advanced Students

We received five correct submissions. We present the solution by Andrea Fanchini.



We have that $\angle CBD = \angle CED$ because both are inscribed in the same arc of circle. Then we have that $BD = CD$ so $\angle BCD = \angle CBD$ so $\angle BCD = \angle CED$.

In the same way, also $\angle DBE = \angle DCE$ because both are inscribed in the same arc of circle.

Now, we note that $\triangle ACD$ and $\triangle CED$ have in common the $\angle ADC$, so $\angle CAD = \angle DCE$. Therefore, $\triangle ACD$ and $\triangle CED$ are similar. Also $\triangle ABD$ and $\triangle BED$ having three equal angles are similar.

So, we have

$$\frac{CE}{CD} = \frac{AC}{AD}, \qquad \frac{BE}{BD} = \frac{AB}{AD}$$

but $BD = CD$, then

$$\frac{CE}{AC} = \frac{BE}{AB}, \qquad \Rightarrow \qquad \frac{BE}{CE} = \frac{AB}{AC}$$

therefore the ratio $\frac{BE}{CE}$ is the same for each circle k.