Markov's Diophantine Equation

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This article describes the story of one Diophantine equation (that is, a polynomial equation in which only integer variables are allowed). The solution uses only the most elementary properties of integers together with a familiar result for quadratic equations, making it accessible to younger audiences. This solution is based on a series of problems, which can be used to run a math circle.

In 1879 in St. Petersburg's University, a 23-year-old scholar defended his Master's thesis entitled "About Binary Quadratic Forms with Positive Determinant." In this thesis, he solved some of the hardest problems in number theory and this work paved the way for future developments of this mathematical area. That young man was Andrei Andreyevich Markov (1856–1922).

The basis for the thesis were two articles published by Markov in Germany in 1879 and 1880 in one of the most famous mathematical journals : *Mathematische Annalen.* Despite this, over 30 years passed before Markov's results were "discovered" by the western community. In 1913, German mathematician Ferdinand Georg Frobenius (1849–1917) published an article "On Markov's numbers." In its preface, he writes that despite Markov's research being "extremely remarkable and important", it is apparently not well known. Frobenius says it is due to the difficulty of exposition as Markov tended to use *continued fractions*, a technique unfashionable at the time.

In this article, we will discuss a Diophantine equation that arose in Markov's research (an equation that now bears his name) :

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

On Diophantine equations

A *Diophantine equation* is an equation which can be written as a polynomial

$$P(x, y, \ldots, w) = 0$$

with integer unknowns $x, y, \ldots, w$ and with integer coefficients.

Diophantine equations often arise naturally. For example, suppose a country makes stamps of denomination 1, 2, 3 and 5 cents ; in how many ways can you make up a postage of n cents ? In solving this problem, you will arrive at the Diophantine equation

$$x + 2y + 3z + 5w = n.$$

Ancient Babylonian mathematicians were interested in constructing right-angle triangles with integer sides, which is equivalent to finding integer solutions to the equation

$$x^2 + y^2 = z^2.$$

The Pythagoreans found a method to construct all the solutions to this equation. And even though the equation might have been found even earlier in Babylon and India, its integer solutions (x, y, z) are called *Pythagorean triples.*

Finding all (integer) solutions to a Diophantine equation, even an easy-looking one, is generally a hard problem. Even for determining whether or not a Diophantine equation has any solutions, it is known that there is no universal method or algorithm. Nowadays, the field of algebraic geometry has developed various techniques to attack specific Diophantine equations. However, we are interested in Markov's Diophantine equation, which can be solved using elementary techniques, so let us come back to it.

Markov's tree

Consider an ordered triple of integers (a, b, c) that is a solution to the given Diophantine equation $P(x, y, z) = 0$. Integers a, b and c are called the *coordinates* of the solution. It is easy to see that if one of the coordinates of the solution of (1) is equal to zero, then all the coordinates are equal to zero ; for that reason, we will only consider solutions without zero coordinates.

The left-hand side of (1) is positive for any solution (a, b, c), so either all of a, b and c are positive or two of them are negative. In the former case, negating any of the two coordinates results in a new solution ; in the latter case, the triple $(|a|, |b|, |c|)$ is also a solution. Therefore, we will only consider solutions (a, b, c) with all positive coordinates.

The symmetry of equation (1) implies that we can form up to six solutions by permuting the coordinates of any given solution ; that is, if (a, b, c) is a solution, we have the six following solutions (not necessarily distinct) :

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Therefore, we will consider all 6 of the above permutations as one solution to Markov's Diophantine equation. As such, only the values of the solution's coordinates are important and not their order.

The equation has one obvious solution, namely $(1, 1, 1)$. We will now see how we can use one solution to find others. Suppose that (a, b, c) is a solution to the Markov's Diophantine equation ; then a is a root of the following quadratic equation :

$$\Phi_a(x) = x^2 + b^2 + c^2 - 3bcx = 0.$$

Then the well-known result for quadratic equations (sometimes also known as Viète's theorem) implies that there is another root $x = a'$ such that

$$a + a' = 3bc, \quad aa' = b^2 + c^2. \tag{2}$$

Clearly, $a' > 0$ and (a', b, c) is also a solution to (1). It is called the *adjacent solution in coordinate* a. Obviously, if (a', b, c) is an adjacent solution to (a, b, c), then (a, b, c) is an adjacent solution to (a', b, c). Similarly, we can define adjacent solutions in coordinates b and c.

Let us find an adjacent solution of $(1, 1, 1)$ in the first coordinate. For that, we need to solve the quadratic equation

$$x^2 + 1^1 + 1^1 - 3 \cdot 1 \cdot 1 \cdot x = 0.$$

This equation has two roots $x = 1$ and $x = 2$. So from the solution $(1, 1, 1)$, we obtain an adjacent solution $(2, 1, 1)$. Following Markov, we shall call these two solutions *singular*. The following important property distinguishes singular solutions from all other solutions.

Exercise 1. Prove that a solution (a, b, c) of Markov's Diophantine equation has two equal coordinates if and only if it is singular.

The first singular solution $(1, 1, 1)$ has only one adjacent solution. The second singular solution $(2, 1, 1)$ has two adjacent solutions : one is $(1, 1, 1)$ and the second one (adjacent in the second coordinate) comes from the equation

$$2^2 + y^2 + 1^2 - 3 \cdot 2 \cdot y \cdot 1 = 0.$$

and is equal to $(2, 5, 1)$. Furthermore, $(2, 5, 1)$ has three adjacent solutions : the previous $(2, 1, 1)$ and two new ones, $(13, 5, 1)$ and $(2, 5, 29)$. In general, any non-singular solution (a, b, c) gives rise to three adjacent solutions

$$(a', b, c), (a, b', c), (a, b, c'),$$

where (compare to (2))

$$a' = 3bc - a, \quad b' = 3ac - b, \quad c' = 3ab - c.$$

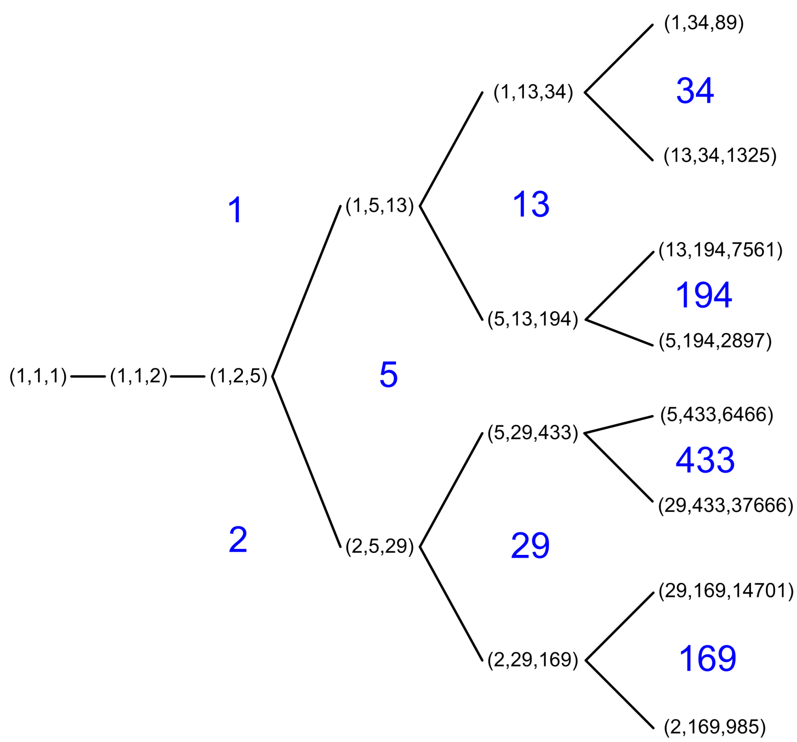
Exercise 2. Suppose that a solution (a, b, c) of Markov's Diophantine equation is non-singular. Prove that the largest coordinate of one of its adjacent solutions is larger than $\max(a, b, c)$ and the largest coordinate of another one of its adjacent solutions is smaller than $\max(a, b, c)$.

Markov's Theorem. Any solution to (1) is connected to the singular solution $(1, 1, 1)$ via a chain of adjacent solutions.

Proof sketch. Let (a, b, c) be a non-singular solution of (1). Use Exercise 2 to deduce that there is a (finite) sequence of adjacent solutions with decreasing maximum coordinate ; hence, we can arrive at a solution (a_n, b_n, c_n) with two equal coordinates. Use Exercise 1. □

Markov's Theorem implies that we can start at a singular solution $(1,1,1)$ and expand to adjacent solutions therefore constructing all possible solutions to Markov's Diophantine equation. This process results in *Markov's tree*, which also allows us for any given $N \geq 1$ to find all solutions to Markov's Diophantine equation whose coordinates are less than N. See the picture below indicating adjacent solutions and values of N.

Exercise 3. Prove that the coordinates of any solution to Markov's Diophantine equation are relatively prime.



Exceptionality of Markov's Diophantine equation

Consider the following question :

> If the sum of the squares of three natural numbers is divisible by their product, what can the quotient be ?

This question is equivalent to the following :

> For which natural numbers k does the Diophantine equation
>
> $$X^2 + Y^2 + Z^2 = kXYZ \qquad (3)$$
>
> have a non-zero solution ?

For $k = 3$, (3) is actually Markov's Diophantine equation. It is easy to see that for $k = 1$ equation (3) has solutions, for example $(3, 3, 3)$. In fact, Hurwitz and Frobenius proved that equation (3) has solutions only for $k = 1$ and $k = 3$. This result can also be derived using elementary methods.

Let us start with the case $k = 1$. We shall see that solving (3) for $k = 1$ reduces to solving Markov's Diophantine equation.

Exercise 4. Let A, B and C be natural numbers and suppose that d of them are not divisible by 3 ($0 \leq d \leq 3$). Prove that the division of $A^2 + B^2 + C^2$ by 3 leaves a remainder of d.

Exercise 5. Prove that all solutions (A, B, C) to the equation

$$X^2 + Y^2 + Z^2 = XYZ \tag{4}$$

are given by

$$A = 3a, \quad B = 3b, \quad C = 3c, \tag{5}$$

where (a, b, c) is a solution to

$$x^2 + y^2 + z^2 = 3xyz. \tag{6}$$

Let us now consider the case $k = 2$.

Exercise 6. Let A, B and C be natural numbers. Prove that the remainder of division of $A^2 + B^2 + C^2$ by 4 is equal to the number of odd integers among A, B and C.

Exercise 7. Prove that equation (3) has no solutions for $k = 2$.

Theorem. Equation (3) has non-zero solutions only for $k = 1$ and $k = 3$.

Proof sketch. Cases $k = 1$ and $k = 2$ are considered above in Exercises 5 and 7. So we have to consider the case $k > 3$. We will prove this case by contradiction; assume that equation (3) has a solution (a, b, c) for some $k > 3$. First, show that the coordinates of the solution are mutually distinct. Next, assume that $a > b > c$ and consider adjacent solutions (a, b, c) and (a', b, c). Using (3), conclude that $a > b > a'$ and hence the largest coordinate of (a', b, c) is less than the largest coordinate of (a, b, c). Proceed similarly to the proof of Markov's theorem by constructing adjacent solutions with smaller and smaller largest coordinates to arrive at a contradiction. □

Corollary. In any solution (a, b, c) to Markov's Diophantine equation all the coordinates are relatively prime.

Proof. Suppose a and b have a common divisor $d > 1$. Then by (1), d also divides c. Hence, we can find natural numbers X, Y and Z such that $a = dX, b = dY, c = dZ$ and $X^2 + Y^2 + Z^2 = 3dXYZ$, which contradicts the above theorem. □

The following equation is an immediate generalization of Markov's Diophantine equation to $n \geq 3$:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = nx_1x_2 \ldots x_n. \tag{7}$$

It is not hard to see that many of the properties of Markov's Diophantine equation proven above also apply to the general case. For example, there exists a singular solution $x_1 = x_2 = \ldots = x_n = 1$; for every solution, there exists an adjacent one; and so on. A general theory of equation (7) constitutes its own research topic.

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Math Quotes

Our federal income tax law defines the tax y to be paid in terms of the income x; it does so in a clumsy enough way by pasting several linear functions together, each valid in another interval or bracket of income. An archeologist who, five thousand years from now, shall unearth some of our income tax returns together with relics of engineering works and mathematical books, will probably date them a couple of centuries earlier, certainly before Galileo and Vieta.

Hermann Weyl in "The Mathematical Way of Thinking", an address given at the Bicentennial Conference at the University of Pennsylvania, 1940.