SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

The editor would like to acknowledge the following solvers whose solutions were overlooked: Arkady Alt for problems 3813, 3815, 3816 and 3818; Paolo Perfetti for 3800, 3815, 3818 and 3820. The editor apologizes sincerely for the oversight.

3831. *Proposed by George Apostolopoulos.*

Let a, b, c, d be positive real numbers with $abcd = 16$. Prove that

$$\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{d^3}+\frac{d^3}{a^3}+4 \geq a+b+c+d\,.$$

Solved by M. Bataille; M. Benito, Ó.Ciaurri, E. Fernández and L. Roncal; O. Geupel; S. Malikić; T. K. Parayiou; P. Perfetti; T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Oliver Geupel; and Titu Zvonaru and Neculai Stanciu.

Each sum is cyclic with four terms. Using the arithmetic-geometric means inequality, we obtain that

$$\begin{aligned}\left[\sum \frac{a^3}{b^3}\right]+4 &= \left[\sum \frac{1}{6}\left(\frac{3a^3}{b^3}+\frac{2b^3}{c^3}+\frac{c^3}{d^3}\right)\right]+4\\ &\geq \left[\sum \frac{a^2}{\sqrt{abcd}}\right]+4 = \left[\sum \frac{a^2}{4}\right]+4\\ &= \frac{1}{4}[a^2+b^2+c^2+d^2+16] = a+b+c+d+\frac{1}{4}\sum(a-2)^2\\ &\geq a+b+c+d,\end{aligned}$$

with equality if and only if $a=b=c=d=2$.

Editor's comment. Perfetti followed a similar strategy. Benito, Ciaurri, Fernández and Roncal applied the arithmetic geometric means inequality to four terms to obtain

$$\frac{3a^3}{b^3}+\frac{2b^3}{c^3}+\frac{c^3}{d^3}+6 \geq \frac{12a}{(abcd)^{1/4}}.$$

This added to its three analogues leads to

$$\sum 6a^3/b^3+24 \geq 12(a+b+c+d)(abcd)^{-1/4}$$

and the result follows.

Zvonaru and Stanciu generalized the method of the solution to obtain

$$\sum_{i=1}^{m} \frac{a_i^{n-1}}{a_{i+1}^{n-1}} + n \geq a_1 + a_2 + \cdots + a_n,$$

where $n \geq 3$, $a_i > 0$ for each i, $a_{n+1} = a_1$ and $a_1 a_2 \dots a_n = 2^n$.

3832. *Proposed by Marcel Chiriţă.*

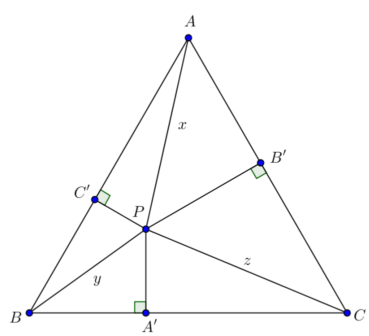
Let P be a point inside the equilateral triangle ABC with side length equal to 1, and let $x = PA$, $y = PB$, $z = PC$. Prove that:

$$(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 = 3(x^2y^2 + y^2z^2 + z^2x^2)\,.$$

Solved by A. Alt; G. Apostolopoulos; Š. Arslanagić; D. Bailey, E. Campbell, and C. Diminnie; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu and T. Zvonaru; M. Benito, Ó. Ciaurri, E. Fernández and L. Roncal; P. De; M. Dinca; O. Kouba; S. Malikić; T. K. Parayiou; P. Perfetti; J. Hawkins and D. R. Stone; D. Văcaru; P. Woo; and the proposer. We present two solutions.

Solution 1, by Peter Woo.

Let PA', PB' and PC' be the perpendiculars from P to BC, CA and AB, respectively. Let $a = PA', b = PB'$ and $c = PC'$.



Since $Q = PA'CB'$ is a cyclic quadrilateral, PC is the diameter of the circumcircle of Q, so by the Law of sines and the Law of cosines we have:

$$\begin{aligned} z = PC &= \frac{A'B'}{\sin \angle A'PB'} = \frac{A'B'}{\sin \frac{2\pi}{3}} \\ &= \frac{2}{\sqrt{3}} A'B' = \frac{2}{\sqrt{3}} \sqrt{a^2 + b^2 - 2ab\cos\frac{2\pi}{3}} = \frac{2}{\sqrt{3}}\sqrt{a^2 + b^2 + ab}. \end{aligned}$$

Hence,

$$z^2 = \frac{4}{3}(a^2 + b^2 + ab)$$

and, similarly,

$$x^2 = \frac{4}{3}(b^2 + c^2 + bc) \text{ and } y^2 = \frac{4}{3}(a^2 + c^2 + ac). \tag{1}$$

It is a well known fact that $a + b + c = \frac{\sqrt{3}}{2}$, the height of $\triangle ABC$. Using this and (1), we have:

$$x^2 - y^2 = \frac{4}{3}(b^2 - a^2 + c(b - a)) = \frac{4}{3}(b - a)(b + a + c) = \frac{2}{\sqrt{3}}(b - a)$$

and, similarly,

$$y^2 - z^2 = \frac{2}{\sqrt{3}}(c - b) \text{ and } z^2 - x^2 = \frac{2}{\sqrt{3}}(a - c). \tag{2}$$

Also,

$$\begin{aligned} x^2 + y^2 + z^2 &= \frac{4}{3}[2(a^2 + b^2 + c^2 + ab + bc + ca)] \\ &= \frac{4}{3}[2(a + b + c)^2 - 3(ab + bc + ca)] \\ &= \frac{4}{3}\left[\frac{3}{2} - 3(ab + bc + ca)\right] = 2 - 4(ab + bc + ca). \end{aligned}$$

Using this and (2), we then have:

$$\begin{aligned} &(x^2 + y^2 + z^2) + (x^2 + y^2 + z^2 - 1)^2 - 3(x^2y^2 + y^2z^2 + z^2x^2) \\ &\qquad = x^4 + y^4 + z^4 - (x^2y^2 + y^2z^2 + z^2x^2) - (x^2 + y^2 + z^2) + 1 \\ &\qquad = \frac{1}{2}\left((x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2\right) - (2 - 4(ab + bc + ca)) + 1 \\ &\qquad = \frac{2}{3}\left((b - a)^2 + (c - b)^2 + (a - c)^2\right) + 4(ab + bc + ca) - 1 \\ &\qquad = \frac{4}{3}\left((a + b + c)^2 - 3(ab + bc + ca)\right) + 4(ab + bc + ca) - 1 \\ &\qquad = \frac{4}{3}\left(\frac{3}{4}\right) - 1 = 0, \end{aligned}$$

which completes the proof.

Solution 2, by M. Benito, Ó.Ciaurri, E. Fernández and L. Roncal slightly expanded by the editor.

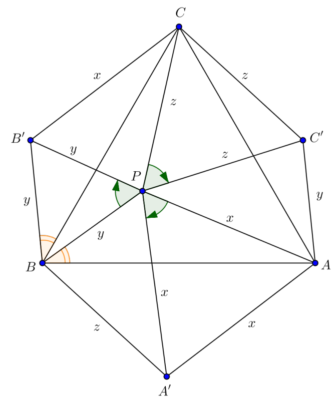
After expanding all the factors, the identity to be proved is equivalent to

$$x^4 + y^4 + z^4 + 1 - (x^2 + y^2 + z^2) + 2(x^2y^2 + y^2z^2 + z^2x^2) = 3(x^2y^2 + y^2z^2 + z^2x^2)$$

or

$$1 + x^4 + y^4 + z^4 = x^2 + y^2 + z^2 + x^2y^2 + y^2z^2 + z^2x^2. \tag{3}$$

To establish (3), we rotate the lines PA, PB, and PC clockwise through an angle of $\pi/3$ radians so A, B, and C are mapped to the points A', B', and C', respectively:



Then clearly, $AA' = x$, $BB' = y$, and $CC' = z$. Since $BC = BA$, $BB' = BP = y$ and $\angle B'BC = \angle PBA = \frac{\pi}{3} - \angle PBC$, we have that $\triangle BB'C \sim \triangle BPA$. Hence, $B'C = PA = x$. Similarly, $C'A = y$ and $A'B = z$.

We now consider the area of the hexagon $H = AA'BB'CC'$. Clearly,

$$\text{Area}(H) = 2 \cdot \text{Area}(\triangle ABC) = 2\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}.$$

On the other hand,

$$\begin{aligned}\text{Area}(H) = &\ \text{Area}(\triangle PAA') + \text{Area}(\triangle PA'B) + \text{Area}(\triangle PBB') \\ &+ \text{Area}(\triangle PB'C) + \text{Area}(\triangle PCC') + \text{Area}(\triangle PC'A).\end{aligned}$$

We also have that

$$\text{Area}(\triangle PAA') = \frac{\sqrt{3}}{4}x^2, \quad \text{Area}(\triangle PBB') = \frac{\sqrt{3}}{4}y^2, \ \text{Area}(\triangle PCC') = \frac{\sqrt{3}}{4}z^2.$$

By Heron's formula, we have $\text{Area}(\triangle PA'B) = \text{Area}(\triangle PB'C) = \text{Area}(\triangle PC'A)$, which equals to

$$\sqrt{\frac{x+y+z}{2} \cdot \frac{x+y-z}{2} \cdot \frac{x-y+z}{2} \cdot \frac{-x+y+z}{2}}.$$

Combining all of the above, we have

$$\begin{aligned}\text{Area}(H) &= \frac{\sqrt{3}}{4}(x^2+y^2+z^2) + \frac{3}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)} \\ &= \frac{\sqrt{3}}{2},\end{aligned}$$

or

$$(x^2+y^2+z^2-2)^2 = 3(x+y+z)(x+y-z)(x-y+z)(-x+y+z),$$

which, after expanding and simplifying, becomes

$$\begin{aligned}&x^4+y^4+z^4+4-4(x^2+y^2+z^2)+2(x^2y^2+y^2z^2+z^2x^2)\\&=3((x+y)^2-z^2)(z^2-(x-y)^2)\\&=3(x^2+y^2-z^2+2xy)(z^2-x^2-y^2+2xy)\\&=-3(x^4+y^4+z^4)+6(x^2y^2+y^2z^2+z^2x^2)\end{aligned}$$

or

$$4(x^4+y^4+z^4)+4=4(x^2+y^2+z^2+x^2y^2+y^2z^2+z^2x^2),$$

which establishes (3) and completes the proof.

Editor's comment. Bailey, Campbell, and Diminnie proved that the result holds for any point P in the plane of A, B, and C. De gave a proof based on the following identity, for which no proof or reference was given:

$$3(x^4+y^4+z^4+s^4)=(x^2+y^2+z^2+s^2)^2,$$

where s denotes the semiperimeter of $\triangle ABC$. If we set $s=1$, then the required identity follows readily after some simple calculations.

It has been pointed out that this is Problem 250 from *Revista Escolar de la Olimpiada Iberoamericana de Mathemática*, whose solutions has been published here: `http://www.oei.es/oim/revistaoim/numero51/250_Bruno.pdf`.

3833. *Proposed by Ángel Plaza.*

Let x, y, z be positive real numbers. Prove that

$$\frac{x^2}{z^3(zx+y^2)}+\frac{y^2}{x^3(xy+z^2)}+\frac{z^2}{y^3(yz+x^2)}\ge\frac{3}{2xyz}\,.$$

Solved by A. Alt; AN-anduud Problem Solving Group; Š. Arslanagić; M. Bataille; D. M. Bătineţu-Giurgiu and N. Stanciu; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. De; O. Geupel; D. Koukakis; S. Malikić; P. McCartney; M. Modak; T. K. Parayiou; P. Perfetti; C. R. Pranesachar; D. Smith; S. Wagon; and the proposer. We present a composite of similar solutions by Arkady Alt; Salem Malikić; Phil McCartney; and Digby Smith.

The claimed inequality is successively equivalent to

$$\sum_{\text{cyclic}}\frac{x^3yz}{z^3\left(zx+y^2\right)}\ge\frac{3}{2}\quad\text{and}\quad\sum_{\text{cyclic}}\frac{\left(\frac{x}{z}\right)^2}{\frac{z}{y}+\frac{y}{x}}\ge\frac{3}{2},$$

which, with the substitutions $a=\frac{x}{z}$, $b=\frac{y}{x}$, $c=\frac{z}{y}$, becomes

$$\sum_{\text{cyclic}}\frac{a^2}{b+c}\ge\frac{3}{2},$$

with $abc = 1$. The Cauchy-Schwarz inequality applied to $\left\langle \frac{a}{\sqrt{b+c}}, \frac{b}{\sqrt{c+a}}, \frac{c}{\sqrt{a+b}} \right\rangle$ and $\left\langle \sqrt{b+c}, \sqrt{c+a}, \sqrt{a+b} \right\rangle$ gives

$$\sqrt{\sum_{\text{cyclic}} \frac{a^2}{b+c}} \cdot \sqrt{2\,(a+b+c)} \geq a+b+c.$$

Dividing by $\sqrt{2\,(a+b+c)}$, squaring, and applying the AM-GM inequality, we have

$$\sum_{\text{cyclic}} \frac{a^2}{b+c} \geq \frac{a+b+c}{2} \geq \frac{3}{2} \cdot \sqrt[3]{abc} = \frac{3}{2},$$

as claimed. Equality holds if and only if $a = b = c$.

Editor's comment. Bătineţu-Girgiu and Stanciu (jointly) indicated that they proposed the same problem to the College Mathematics Journal and then School Science and Mathematics. Stan Wagon provided a Mathematica solution.

3834. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a parallelogram and E, F be interior points of the sides BC, CD, respectively, such that $\frac{BE}{EC} = \frac{CF}{FD}$. The line segments AE and AF meet the diagonal BD at the points K and L respectively.

(a) Prove that $\text{Area}(\Delta AKL) = \text{Area}(\Delta BKE) + \text{Area}(\Delta DLF)$.

(b) Find the ratio $\dfrac{\text{Area}(\Delta ABCD)}{\text{Area}(\Delta AECF)}$.

Solved by M. Amengual Covas (2 solutions); AN-anduud Problem Solving Group; M. Bataille; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. De; P. Deiermann; O. Kouba; T. K. Parayiou; R. Peiró; P. Y. Woo; and the proposer. We present three solutions. In each, we use $[ABC]$ to denote the area of ABC.

Solution 1, by Prithwijit De.

Let $\dfrac{BE}{EC} = \dfrac{CF}{FD} = x$. Observe that

$$\frac{[ABE]}{[ABC]} = \frac{BE}{BC} = \frac{x}{x+1} \quad \text{and} \quad \frac{[AFD]}{[ACD]} = \frac{DF}{CD} = \frac{1}{x+1}.$$

But $[ABC] = [ACD] = \dfrac{1}{2}[ABCD]$. Therefore

$$[ABE] + [AFD] = \frac{1}{2}[ABCD] = [ABD]. \tag{1}$$

Also

$$[ABE] = [BKE] + [AKB],$$

$$[AFD] = [ALD] + [DLF]$$

and

$$[ABD] = [AKB] + [AKL] + [ALD].$$

Using this in the above equation we get

$$[AKL] = [BKE] + [DLF].$$

This proves part (a). For part (b), observe that

$$[ABE] + [AFD] + [AECF] = [ABCD]. \qquad (2)$$

Using (1) and (2), we obtain $\dfrac{[ABCD]}{[AECF]} = 2$.

Solution 2, by Omran Kouba.

Let $\vec{i} = \vec{AB}$ and $\vec{j} = \vec{AD}$. Since E is an interior point of the side BC, we have $\vec{BE} = \alpha\vec{j}$ for some $\alpha \in (0,1)$. Now the fact that $\frac{BE}{EC} = \frac{CF}{FD}$ implies that $\vec{DF} = (1-\alpha)\vec{i}$. Now the equation of the line BD in the affine coordinate system $(A; \vec{i}, \vec{j})$ is $x + y = 1$. So, the point K defined by $\vec{AK} = t\vec{AE} = t\vec{i} + t\alpha\vec{j}$ belongs to BD if and only if $t(1+\alpha) = 1$. Hence

$$\vec{AK} = \left(\frac{1}{1+\alpha}\right)\vec{i} + \left(\frac{\alpha}{1+\alpha}\right)\vec{j}.$$

Similarly, we find that

$$\vec{AL} = \left(\frac{1-\alpha}{2-\alpha}\right)\vec{i} + \left(\frac{1}{2-\alpha}\right)\vec{j}.$$

Now,

$$2\,[AKL] = \det(\vec{AK}, \vec{AL}) = \frac{1-\alpha+\alpha^2}{(1+\alpha)(2-\alpha)} \cdot \det(\vec{i}, \vec{j})$$

$$2\,[BKE] = \det(\vec{BE}, \vec{BK}) = \det\left(\alpha\vec{j}, \frac{\alpha}{1+\alpha}(\vec{j} - \vec{i})\right) = \frac{\alpha^2}{1+\alpha} \cdot \det(\vec{i}, \vec{j})$$

$$2\,[DLF] = \det(\vec{DL}, \vec{DF}) = \det\left(\frac{1-\alpha}{2-\alpha}(\vec{i} - \vec{j}), (1-\alpha)\vec{j}\right) = \frac{(1-\alpha)^2}{2-\alpha} \cdot \det(\vec{i}, \vec{j}).$$

Thus, (a) follows from the fact that

$$\frac{\alpha^2}{1+\alpha} + \frac{(1-\alpha)^2}{2-\alpha} = \frac{1-\alpha+\alpha^2}{(1+\alpha)(2-\alpha)}.$$

On the other hand,

$$\begin{aligned}
[AECF] &= [ABCD] - [ABE] - [AFD] \\
&= [ABCD] - \frac{1}{2}\det(\vec{AB}, \vec{BE}) - \frac{1}{2}\det(\vec{DF}, \vec{AD}) \\
&= [ABCD] - \frac{1}{2}\det(\vec{i}, \alpha\vec{j}) - \frac{1}{2}\det((1-\alpha)\vec{i}, \vec{j}) \\
&= [ABCD] - \frac{1}{2}\det(\vec{i}, \vec{j}) = \frac{1}{2}\,[ABCD]
\end{aligned}$$

because $[ABCD] = \det(\vec{i}, \vec{j})$. Thus, $\dfrac{[ABCD]}{[AECF]} = 2$, which is the answer to (b).

Solution 3, by Miguel Amengual Covas.

We use analytic geometry. We let the axes be $Ox \equiv BC$ and $Oy \equiv BA$ and give coordinates $B\,(0,0)$, $C\,(a,0)$ and $A\,(0,b)$.

We put $\frac{BE}{EC} = \frac{CF}{FD} = k$. Then the coordinates of D are (a,b), the coordinates of E are $\left(\frac{ka}{k+1}, 0\right)$, and those of F are $\left(a, \frac{kb}{k+1}\right)$. Hence, the coordinates of K are $\left(\frac{ka}{2k+1}, \frac{kb}{2k+1}\right)$ and the coordinates of L are $\left(\frac{(k+1)a}{k+2}, \frac{(k+1)b}{k+2}\right)$.

We put $\angle ABC = \theta$. Since the area of a triangle whose vertices have coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), is given by

$$\frac{\sin\theta}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

provided the three vertices are listed in counterclockwise order around the triangle, we get

$$[AKL] = \frac{\sin\theta}{2}\begin{vmatrix} 0 & b & 1 \\ \frac{ka}{2k+1} & \frac{kb}{2k+1} & 1 \\ \frac{(k+1)a}{k+2} & \frac{(k+1)b}{k+2} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2}\cdot\frac{k^2+k+1}{(k+2)\,(2k+1)}\ ,$$

$$[BKE] = \frac{\sin\theta}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{ka}{k+1} & 0 & 1 \\ \frac{ka}{2k+1} & \frac{kb}{2k+1} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2}\cdot\frac{k^2}{(k+1)\,(2k+1)}\ ,$$

and

$$[DLF] = \frac{\sin\theta}{2}\begin{vmatrix} a & b & 1 \\ \frac{(k+1)a}{k+2} & \frac{(k+1)b}{k+2} & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} = \frac{ab\sin\theta}{2}\cdot\frac{1}{(k+1)\,(k+2)}$$

Hence,

$$\begin{aligned}[BKE] + [DLF] &= \frac{ab\sin\theta}{2(k+1)}\left(\frac{k^2}{2k+1} + \frac{1}{k+2}\right) = \frac{ab\sin\theta}{2\,(k+1)}\cdot\frac{k^2\,(k+2)+2k+1}{(2k+1)\,(k+2)} \\ &= \frac{ab\sin\theta}{2\,(k+1)}\cdot\frac{(k+1)\left(k^2+k+1\right)}{(2k+1)\,(k+2)} = \frac{ab\sin\theta}{2}\cdot\frac{k^2+k+1}{(2k+1)\,(k+2)} \\ &= S\,[AKL]\end{aligned}$$

as desired. For part (b), we have

$$\begin{aligned}[AECF] &= [AEF]+[FEC] \\ &= \tfrac{\sin\theta}{2}\left(\begin{vmatrix} 0 & b & 1 \\ \frac{ka}{k+1} & 0 & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} + \begin{vmatrix} \frac{ka}{k+1} & 0 & 1 \\ a & 0 & 1 \\ a & \frac{kb}{k+1} & 1 \end{vmatrix} \right) \\ &= \tfrac{ab\sin\theta}{2}\left(\tfrac{k^2+k+1}{(k+1)^2}+\tfrac{k}{(k+1)^2}\right) \\ &= \tfrac{1}{2}ab\sin\theta \\ &= \tfrac{1}{2}[ABCD]\end{aligned}$$

that is,

$$\frac{[ABCD]}{[AECF]}=2.$$

3835. *Proposed by Marcel Chiriţă, Bucharest, Romania.*

Determine the functions $f:\mathbb{R}\to\mathbb{R}$, continuous at $x=0$, for which $f(0)=1$ and

$$3f(x)-5f(\alpha x)+2f(\alpha^2 x)=x^2+x\,,$$

for all $x\in\mathbb{R}$, where $\alpha\in(0,1)$ is fixed.

Solved by A. Alt; AN-anduud Problem Solving Group; M. Bataille; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; O. Geupel; O. Kouba; D. Koukakis; J. Ling; S. de Luxán and Á. Plaza; S. Muralidharan; and the proposer. We present 2 solutions.

Solution 1, by many of the solvers.

The required functions are

$$f(x)=\frac{x^2}{(1-\alpha^2)(3-2\alpha^2)}+\frac{x}{(1-\alpha)(3-2\alpha)}+1.$$

Of course, computation shows that the given functions satisfy the conditions. Let us see that they are the only possible ones.

By the condition on f, we have

$$\begin{aligned}\sum_{k=0}^{n}\left((\alpha^k x)^2+\alpha^k x\right) &= \sum_{k=0}^{n}\left(3f(\alpha^k x)-5f(\alpha^{k+1}x)+2f(\alpha^{k+2}x)\right) \\ &= 3f(x)-2f(\alpha x)-3f(\alpha^{n+1}x)+2f(\alpha^{n+2}x).\end{aligned}$$

Then,

$$\lim_{n\to\infty}\left(3f(x)-2f(\alpha x)-3f(\alpha^{n+1}x)+2f(\alpha^{n+2}x)\right)=x^2\sum_{k=0}^{\infty}\alpha^{2k}+x\sum_{k=0}^{n}\alpha^k$$

and so

$$3f(x)-2f(\alpha x)-1=\frac{x^2}{1-\alpha^2}+\frac{x}{1-\alpha},$$

because $\lim_{n\to\infty} f(x_n)=1$ for any sequence such that $\lim_{n\to\infty} x_n=0$. Using this equation, we obtain that

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k\left(\frac{(\alpha^k x)^2}{1-\alpha^2}+\frac{\alpha x}{1-\alpha}+1\right)&=\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k\left(3f(\alpha^k x)-2f(\alpha^{k+1}x)\right)\\&=3f(x)-2\left(\frac{2}{3}\right)^n f(\alpha^{n+1}x).\end{aligned}$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\left(3f(x)-2\left(\frac{2}{3}\right)^n f(\alpha^{n+1}x)\right)\\&\qquad=\frac{x^2}{1-\alpha^2}\sum_{k=0}^{\infty}\left(\frac{2\alpha^2}{3}\right)^k+\frac{x}{1-\alpha}\sum_{k=0}^{\infty}\left(\frac{2\alpha}{3}\right)^k+\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k,\end{aligned}$$

which yields, finally,

$$f(x)=\frac{x^2}{(1-\alpha^2)(3-2\alpha^2)}+\frac{x}{(1-\alpha)(3-2\alpha)}+1.$$

Solution 2, by Omran Kouba.

Consider a function f satisfying the proposed conditions. Since

$$3-5\alpha+2\alpha^2=(3-2\alpha)(1-\alpha)\neq 0\quad\text{and}\quad 3-5\alpha^2+2\alpha^4\neq 0,$$

we can define $g:\mathbb{R}\to\mathbb{R}$ by the formula

$$g(x)=f(x)-\frac{x}{3-5\alpha+2\alpha^2}-\frac{x^2}{3-5\alpha^2+2\alpha^4}.$$

Clearly, g is continuous at $x=0$ with $g(0)=1$, and it is straightforward to check that

$$3g(x)-5g(\alpha x)+2g(\alpha^2 x)=0,$$

for all real numbers x.

Now, let t be a fixed nonzero real number, and consider the sequence $\{u_n\}$ defined by $u_n=g(\alpha^n t)$. Using the above relation for g with $x=\alpha^n t$ we see that

$$2u_{n+2}-5u_{n+1}+3u_n=0,\qquad\text{for every } n\geq 0.$$

The characteristic polynomial associated with this linear recursive sequence is $2\lambda^2-5\lambda+3$ and it has two real zeros: 1 and 3/2. This proves that $u_n = a+b\left(\frac{3}{2}\right)^n$ for some constants a and b. The continuity of g at 0 implies that $\lim_{n\to\infty} u_n = g(0)$, so $b = 0$, and the fact that $g(0) = 1$ implies that $a = 1$. Thus $u_n = 1$ for every n. In particular, $g(t) = u_0 = 1$, but t is arbitrary, so $g \equiv 1$. Going back to the definition of g, we see that, for every $x \in \mathbb{R}$, we have

$$f(x) = 1 + \frac{x}{3-5\alpha+2\alpha^2} + \frac{x^2}{3-5\alpha^2+2\alpha^4}.$$

Conversely, it is readily seen that any function f of this form satisfies the conditions of the problem. So, these are all the solutions of the proposed problem.

Editor's comment. Some solutions featured parts of both methods shown above. It has been remarked that this problem appeared in *Mathematics Magazine*, 87 (1), February 2014, as problem 1939 as well as Problem 239 in *Revista Escolar de la Olimpiada Iberoamericana de Mathemática.*

3836. *Proposed by Jung In Lee.*

Determine all triplets (a, b, c) of positive integers that satisfy

$$a! + b^b = c!$$

Solved by B. Beasley; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; S. Malikić; D. Stone and J. Hawkins; and the proposer. We present two solutions.

Solution 1, by the proposer Jung In Lee.

Since $a! < c!$ and $b^b < c! \le c^c$, then $a < c$ and $b < c$. Suppose that $a \le 2$. Since b divides $c!$, it divides $a!$ so that $b \le 2$. The only possible solutions in this situation are $(a, b, c) = (1, 1, 2), (2, 2, 3)$. Henceforth, assume that $a \ge 3$.

With $v_2(x)$ denoting the exponent of the largest power of 2 dividing x, we find that $b \le v_2(b^b) = v_2(a!)$ since $b^b = a![(a+1)\ldots(c-1)c-1]$, the second factor being odd. Thus

$$b \le v_2(a!) = \sum_{k=1}^{\infty}\left\lfloor\frac{a}{2^k}\right\rfloor < \sum_{k=1}^{\infty}\frac{a}{2^k} = a.$$

Let $\{2 = p_1, 3 = p_2, p_3, \cdots, p_n, \cdots\}$ be the sequence of all primes and suppose that $p_k \le a < p_{k+1}$ so that $p_k \ge 3$. Since $a!$ divides b^b, then b is divisible by $p_1p_2\ldots p_k$. Since $p_1p_2\cdots p_k - 1$ exceeds 1 and is divisible by none of the first k primes, it is not less than p_{k+1}. Hence

$$p_{k+1} > a > b > p_1p_2\cdots p_k - 1 \ge p_{k+1},$$

a contradiction. Hence $a \le 2$ and we have found all the solutions.

Editor's comment. The above proposer's solution is the only solution that did not make use of Bertrand's postulate.

Solution 2, by Joseph DiMuro, expanded by the editor.

Note that $c > a$ and $c > b$ since $c! > b^b > b!$. Suppose that p is a prime divisor of b. Then p must divide $b!$, b^b and $c!$, so that p must divide $a!$ and $p \leq a$. Thus, if $a = 1$, then $b = 1$ and we get the solution $(a, b, c) = (1, 1, 2)$. If $a = 2$, then $b \neq 1$ and the only prime divisor of b is 2. But then b^b is a multiple of 4 and $c! \equiv 2 \pmod 4$. The only possibility is $(a, b, c) = (2, 2, 3)$.

Suppose, if possible, that $a \geq 3$; let q be the largest prime that does not exceed a. Then, by Bertrand's postulate that when $m \geq 2$ there is always a prime between m and $2m$, $a < 2q$, so that q^2 cannot divide $a!$. However, since q divides a, it must divide $c!$ and hence divide b. Since $a \geq 3$, $a!$ and $c!$ are both even as is b. Because $b \neq 2$, we must have that $c \geq b \geq 2q$ (whether $q = 2$ or q is odd). Hence q^2 divides $c!$ and b^b and so must divide $a!$, yielding a contradiction.

Therefore, the sole solutions are $(a, b, c) = (1, 1, 2), (2, 2, 3)$.

3837. *Proposed by Arkady Alt.*

Let $(u_n)_{n\geq 0}$ be a sequence defined recursively by

$$u_{n+1} = \frac{u_n + u_{n-1} + u_{n-2} + u_{n-3}}{4},$$

for $n \geq 3$. Determine $\lim_{n\to\infty} u_n$ in terms of u_0, u_1, u_2, u_3.

Solved by AN-anduud Problem Solving Group; R. Barbara; M. Bataille; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; P. Deiermann; J. DiMuro; O. Kouba; K. Lewis; Á. Plaza; C. R. Pranesachar; D. Smith; D. Stone and J. Hawkins; R. Zarnowski; and the proposer. We present 2 solutions.

Solution 1, by Joseph DiMuro.

We prove that $\lim_{n\to\infty} u_n = \frac{1}{10}u_0 + \frac{2}{10}u_1 + \frac{3}{10}u_2 + \frac{4}{10}u_3$. The proof is by visual aid.

Put 10 water glasses on a table. Pour u_0 mL of water into one glass. Pour u_1 mL of water into each of 2 glasses, pour u_2 mL into each of 3 glasses, and pour u_3 mL into each of the remaining 4 glasses. Put the glasses into groups based on the amount of water in each glass. (So, the lone glass with u_0 mL is in a group by itself, the 2 glasses with u_1 mL form another group, and so on.)

Now, perform the following operation repeatedly: take one glass from each group. Pour water between those four glasses until they all have the same amount. Then put those four glasses back on the table as a new group. (Each of the old groups will have one fewer glass than before.)

After performing this operation once, you will have 1 glass with u_1 mL, 2 glasses with u_2 mL, 3 glasses with u_3 mL, and 4 glasses with u_4 mL. After performing this operation a second time, you will have 1 glass with u_2 mL, 2 glasses with u_3 mL, 3 glasses with u_4 mL, and 4 glasses with u_5 mL. And so on.

The amount of water in each glass will gradually approach $\lim\limits_{n\to\infty} u_n$. Therefore, $\lim\limits_{n\to\infty} u_n$ must be equal to the average amount of water per glass at the start:

$$\lim_{n\to\infty} u_n = \frac{1}{10}u_0 + \frac{2}{10}u_1 + \frac{3}{10}u_2 + \frac{4}{10}u_3.$$

Editor's comment. This solution, as well the argument above, generalize to any sequence where u_n is defined to be the average of the k previous terms. This solution does assume that $\lim_{n\to\infty} u_n$ exists. Its existence can be proven using the roots of the characteristic polynomial for the recurrence relation, as in the next solution.

Solution 2, by Michel Bataille.

The characteristic equation of the sequence $(u_n)_{n\geq 0}$ is

$$4x^4 - x^3 - x^2 - x - 1 = 0,$$

that is,

$$(x-1)(4x^3 + 3x^2 + 2x + 1) = 0.$$

The function $f : x \mapsto 4x^3 + 3x^2 + 2x + 1$ is continuous and strictly increasing on $\mathbb{R}$ with $f(\mathbb{R}) = \mathbb{R}$, so the equation $f(x) = 0$ has a unique real solution, say r. Noticing that $f(-1) = -2 < 0$ and $f(-\frac{1}{4}) > 0$, we see that

$$-1 < r < -\frac{1}{4} \qquad (1)$$

.

The non real solutions to $f(x) = 0$ are two complex conjugates z_0 and $\overline{z_0}$ and since $r \cdot z_0 \cdot \overline{z_0} = -\frac{1}{4}$, we have $|z_0|^2 = \frac{1}{4|r|}$, hence $|z_0| < 1$ since by (1), $|r| > \frac{1}{4}$.

From the list $1, r, z_0, \overline{z_0}$ of the roots of the characteristic equation, we deduce the form of u_n:

$$u_n = \alpha_1 + \alpha_2 r^n + \alpha_3 z_0^n + \alpha_4 \overline{z_0}^n$$

where the α_j are independent of n and determined from u_0, u_1, u_2, u_3.

Since $|r| < 1$ and $|z_0| = |\overline{z_0}| < 1$, we have $\lim_{n\to\infty} r^n = \lim_{n\to\infty} z_0^n = \lim_{n\to\infty} \overline{z_0}^n = 0$ so that $\lim\limits_{n\to\infty} u_n = \alpha_1$.

Now, the following relations hold

$$u_0 = \alpha_1+\alpha_2+\alpha_3+\alpha_4, \quad u_1 = \alpha_1+\alpha_2 r+\alpha_3 z_0+\alpha_4\overline{z_0}, \quad u_2 = \alpha_1+\alpha_2 r^2+\alpha_3 z_0^2+\alpha_4\overline{z_0}^2$$

and

$$u_3 = \alpha_1 + \alpha_2 r^3 + \alpha_3 z_0^3 + \alpha_4 \overline{z_0}^3.$$

Since $f(r) = f(z_0) = f(\overline{z_0}) = 0$, we obtain

$$u_0 + 2u_1 + 3u_2 + 4u_3 = 10\alpha_1 + \alpha_2 f(r) + \alpha_3 f(z_0) + \alpha_4 f(\overline{z_0}) = 10\alpha_1$$

and we conclude
$$\lim_{n\to\infty} u_n = \frac{u_0 + 2u_1 + 3u_2 + 4u_3}{10}.$$

Editor's comment. Perfetti pointed out that this result appeared in *"On the Solutions of Linear Mean Recurrences"*, American Mathematical Monthly, 121 (6).

3838. *Proposed by Jung In Lee.*

Prove that there are no triplets (a, b, c) of distinct positive integers that satisfy the conditions:

- $a + b$ divides c^2, $b + c$ divides a^2, $c + a$ divides b^2, and
- the number of distinct prime factors of abc is at most 2.

Solved by M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; S. Malikić; and the proposer. We present the proposer's solution.

Suppose (a, b, c) is a triplet of distinct positive integers satisfying the given conditions. Let $a = p^{x_1}q^{y_1}$, $b = p^{x_2}q^{y_2}$ and $c = p^{x_3}q^{y_3}$, where p and q are distinct prime numbers and x_i and y_i are nonnegative integers for $i = 1, 2, 3$. Let $i, j, k \in \{1, 2, 3\}$ such that $i \neq j \neq k \neq i$. We consider two cases separately.

Case 1. Suppose $x_i > x_j$ and $y_i > y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j}q^{y_i-y_j} + 1) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j}q^{y_i-y_j} + 1 | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j}q^{y_i-y_j} + 1, p^{2x_k}q^{2y_k}) = 1$.

Case 2. Suppose $x_i > x_j$ and $y_i < y_j$. Then we have
$$p^{x_j}q^{y_j}(p^{x_i-x_j} + q^{y_j-y_i}) = p^{x_i}q^{y_i} + p^{x_j}q^{y_j},$$
which divides $p^{2x_k}q^{2y_k}$. So
$$p^{x_i-x_j} + q^{y_j-y_i} | p^{2x_k}q^{2y_k},$$
which is impossible since $(p^{x_i-x_j} + q^{y_j-y_i}, p^{2x_k}q^{2y_k}) = 1$.

By cases 1 and 2, we have $x_i = x_j$ or $y_i = y_j$. It follows that either two or more of the statements $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_1$ are true or two or more of the statements $y_1 = y_2$, $y_2 = y_3$ and $y_3 = y_1$ are true. Hence $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$. Without loss of generality, we assume that $x_1 = x_2 = x_3 = x$. Since the given conditions are homogenous in a, b and c, which are distinct, we may assume that $y_1 > y_2 > y_3$. Then
$$p^x q^{y_1} + p^x q^{y_2} = p^x q^{y_2}(q^{y_1-y_2} + 1),$$

which divides $p^{2x}q^{y_3}$, so $q^{y_1-y_2}+1=p^l$ for some integer $l>0$. Similarly, $q^{y_2-y_3}+1=p^m$ and $q^{y_1-y_3}+1=p^n$ for some positive integers m and n.

Hence $p^n=q^{y_1-y_3}+1<(q^{y_1-y_2}+1)(q^{y_2-y_3}+1)=p^{l+m}$, so $n+1\le l+m$ and $2p^n\le p^{l+m}$. But

$$\begin{aligned}2p^n-p^{l+m}&=2q^{y_1-y_3}+2-(q^{y_1-y_2}+1)(q^{y_2-y_3}+1)\\&=q^{y_1-y_3}+1-(q^{y_1-y_2}+q^{y_2-y_3})\\&=(q^{y_1-y_2}-1)(q^{y_2-y_3}-1)>0\end{aligned}$$

a contradiction and the proof is complete.

Editor's comment. DiMuro gave the triplet

$$(a,b,c)=(90,180,720)=(2\times 3^2\times 5, 2^2\times 3^2\times 5, 2^4\times 3^2\times 5)$$

as an example of three distinct positive integers satisfying the first condition such that abc has three distinct prime factors.

This problem is very similar to OC95, which originally appeared with a typo.

3839. *Proposed by Peter Y. Woo.*

Let ΔABC be an acute triangle, and P any point on the plane. Let AD, BE, CF be the altitudes of ΔABC. Let D', E', F' be the circumcentres of ΔPAD, ΔPBE, ΔPCF respectively. Prove that D', E', F' are collinear.

Solved by M. Bataille; R. Barroso Campos; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; S. Malikić; T. K. Parayiou; N. Stanciu and T. Zvonaru; and the proposer. We present the solution by Ricardo Barroso Campos.

As usual we denote the orthocentre of ΔABC by H. Using similar right triangles, one sees that

$$AH\cdot HD=BH\cdot HE=CH\cdot HF.$$

Let PH intersect the circumcircle of ΔPAD again at Q. Then

$$PH\cdot HQ=AH\cdot HD,$$

which implies that the other two circumcircles, of ΔPBE and ΔPCF, also pass through Q. We conclude that the circumcentres D',E',F' all lie on the perpendicular bisector of the common chord PQ.

Editor's comments. Malikić pointed out that should P be chosen on an altitude or its extension, then a more careful statement of the problem would be required to give rise to a meaningful result. On the other hand, it is clear from the featured solution that there was no need to require ΔABC to be acute. This observation was provided by Bataille and by Benito et al. The latter group observed, moreover, that the point Q (in the featured solution) is known as the *orthoassociate of* P, which is the point $X(5523)$ in Clark Kimberling's *Encyclopedia of Triangle Centers.* The point is discussed further in Bernard Gibert's "Orthocorrespondence and Orthopivotal Cubics," *Forum Geometricorum* **3** (2003), 1–27. Stanciu and Zvonaru noted that in the special case where P is the centroid of ΔABC, then Q is Kimberling's point $X(468)$.

3840$\star$. *Proposed by Šefket Arslanagić.*

Prove or disprove

$$a^3c + ab^3 + bc^3 \geq a^2b^2 + b^2c^2 + c^2a^2 \,,$$

where $a, b, c > 0$.

Solved by A. Alt; AN-anduud Problem Solving Group; R. Barbara; M. Bataille; D. Bailey, E. Campbell, and C. Diminnie; D. M. Bătineţu-Giurgiu, N. Stanciu, and T. Zvonaru; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; J. DiMuro; O. Geupel; O. Kouba; S. Malikić; P. Perfetti; C. R. Pranesachar; H. Ricardo; D. Smith; S. Wagon; and P. Y. Woo.

All the solvers disproved the given inequality by giving various counterexamples, some of which are $(a, b, c) = (1, 6, 4), (1, 7, 4), (1, 8, 4), (1, 20, 10), (1, 10, 5), (5, 2, 0.5)$. Barbara showed that the inequality is false even if the right-hand side is replaced by $\epsilon(a^2b^2 + b^2c^2 + c^2a^2)$ where $\epsilon > 0$ is arbitrary. Benito, Ciaurri, Fernández and Roncal, showed that $(a, 12a, 6a)$ provides counterexamples for all $a > 0$. Kouba, and Bailey, Campbell and Diminnie pointed out that the symmetric version holds since

$$a^3b+ab^3+b^3c+bc^3+c^3a+ca^3-2(a^2b^2+b^2c^2+c^2a^2) = ab(a-b)^2+bc(b-c)^2+ca(c-a)^2$$

is greater or equal than zero. Malikić and Pranesachar proved that the inequality holds if a, b and c are the side lengths of a triangle. Pranesachar also commented that the inequality holds if $a \leq b \leq c$ or $b \leq c \leq a$ or $c \leq a \leq b$. Bailey, Campbell, and Diminnie actually gave a proof for this as well as the fact that if $a \leq c \leq b$, $c \leq b \leq a$, or $b \leq a \leq c$, then $a^2b^2 + b^2c^2 + c^2a^2 \leq a^3b + b^3c + c^3a$.