OLYMPIAD SOLUTIONS

OC116. Find all positive integers n which are 300 times the sum of their digits.

Originally question 2 from Italian Math Olympiad 2012.

Solved by O. Geupel, R. Hess; S. Muralidharan; D. Văcaru; T. Zvonaru and N. Stanciu; and K. Zelator. We give the solution by S. Muralidharan.

We show that 2700 is the only number with the property.

Let N be a $k-$digit number such that N equals 300 times the sum of its digits. As N is divisible by 300, the last two digits must be 00, and hence the sum of the digits of N is at most $9(k-2)$. Thus we must have

$$10^{k-1} < N = 300 \times \text{sum of digits} \leq 300 \times 9(k-2)\,.$$

The above inequality yields $k \leq 4$. Clearly $k \geq 3$.

If $k=3$, then, as 300 divides N, we must have $N = 300, 600$ or 900 and it is easy to see that none of these values work.

Let $k=4$. Then $N = \overline{a_1a_200} = 1000a_1 + 100a_2$. Thus we must have $10a_1 + a_2 = 3(a_1 + a_2)$ and hence $7a_1 = 2a_2$. The only possibility is $a_1 = 2$ and $a_2 = 7$. Thus 2700 is the only $4-$digit number with the required property.

OC117. Find the smallest positive integer m such that for all prime numbers $p > 3$, we have

$$105 | 9^{p^2} - 29^p + m\,.$$

Originally question 1 from China Western Olympiad 2012.

Solved by N. Evgenidis; O. Geupel; D. E. Manes; D. Văcaru; K. Zelator. We give the solution of Nikolaos Evgenidis.

Since $105 = 3 \cdot 5 \cdot 7$ we must have

$$9^{p^2} - 29^p + m \equiv 0 \pmod{3, 5, 7}\ .$$

Since $3 \mid 9^{p^2}$, we have

$$0 \equiv -29^p + m \equiv -(-1)^p + m \equiv 1 + m \pmod 3\ .$$

This shows that $m \equiv -1 \pmod 3$.

Modulo 5 we also have

$$0 \equiv 9^{p^2} - 29^p + m \equiv (-1)^{p^2} - (-1)^p_m \equiv m \pmod 5\ .$$

This shows that $m \equiv 0 \pmod 5$.

Finally, modulo 7 we have

$$0 \equiv 9^{p^2} - 29^p + m \equiv 2^{p^2} - 1^p + m \equiv 2^{p^2} - 1 + m \pmod 7 .$$

As $p^2 \equiv 1 \pmod 3$ and $2^3 \equiv 1 \pmod 7$ we have $2^{p^2} \equiv 2 \pmod 7$.

Therefore

$$0 \equiv 2 - 1 + m = m + 1 \pmod 7 .$$

Now $m \equiv -1 \pmod{3, 7}$ is equivalent to $m \equiv -1 \equiv 20 \pmod{21}$. Also, we know that we must have $m \equiv 0 \equiv 20 \pmod 5$. By the Chinese Remainder Theorem, this is equivalent to

$$m \equiv 20 \pmod{105} .$$

The smallest m is therefore $m = 20$.

OC118. Find all functions $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfying

$$f(x) + f(y) \leq \frac{f(x+y)}{2} \qquad \text{and} \qquad \frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y} ,$$

for all $x, y \in (0, \infty)$.

Originally question 4 from Indian IMO training camp 2012, day 3.

Solved by Oliver Geupel whose solution we present below.

It is straightforward to verify that the functions $f(x) = cx^2$ with $c \leq 0$ are solutions of the problem. We prove that there are no other solutions.

Let f be any solution and write $g(x) = f(x)/x$. The inequalities rewrite as

$$2(xg(x) + yg(y)) \leq (x+y)g(x+y) , \tag{1}$$

$$g(x+y) \leq g(x) + g(y) . \tag{2}$$

By (1) and (2),

$$2(xg(x) + yg(y)) \leq (x+y)g(x+y) \leq (x+y)(g(x) + g(y)) , \tag{3}$$

whence $(x-y)(g(x) - g(y)) \leq 0$, that is, the function g is nonincreasing.

Setting $x = y$ in (3), we obtain $g(2x) = 2g(x)$. A straightforward inductive argument yields $g(2^n x) = 2^n g(x)$ for every natural number n.

We prove by induction on n that $g(nx) = ng(x)$. We have seen that the assertion holds for n being a power of 2. Assume that $n = 2^q + r$ where $0 \leq r < 2^q$ and that for every $k < n$ it holds $g(kx) = kg(x)$.

By induction hypothesis and (2), it holds

$$\begin{aligned} 2^{q+1}g(x) &= g(2^{q+1}x) \leq g((2^q + r)x) + g((2^q - r)x) = g(nx) + (2^q - r)g(x) \\ &\leq (2^q + r)g(x) + (2^q - r)g(x) = 2^{q+1}g(x) . \end{aligned}$$

Thus, $g(nx) + (2^q - r)g(x) = 2^{q+1}g(x)$, that is, $g(nx) = ng(x)$, which completes the induction.

We deduce that

$$g\left(\frac{m}{n}\right) = mg\left(\frac{1}{n}\right) = m \cdot \frac{1}{n}g(1) = \frac{m}{n}g(1)$$

holds for natural numbers m and n. Thus, for every positive rational number x it holds that $g(x) = x \cdot g(1)$. Since g is nonincreasing, we have $g(1) \leq 0$.

Finally, let x be any positive real number. Let (y_k) and (z_k) be sequences of rational numbers such that $y_k < x < z_k$ for $k = 1, 2, 3, \ldots$ as well as

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k = x.$$

Since g is nonincreasing, we have $g(z_k) \leq g(x) \leq g(y_k)$. Because

$$\lim_{k\to\infty} g(y_k) = \lim_{k\to\infty} g(z_k) = x \cdot g(1),$$

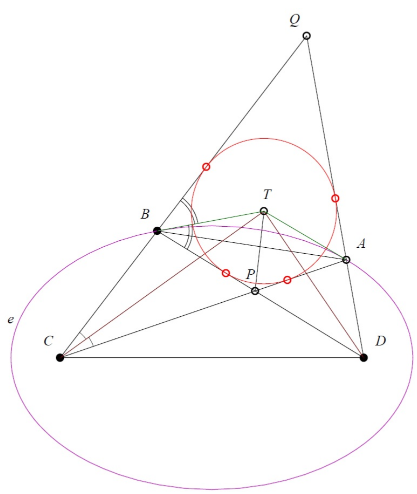
we deduce $g(x) = x \cdot g(1)$. Consequently, $f(x) = x \cdot g(x) = x^2 \cdot g(1)$.

OC119. Let $ABCD$ be a convex quadrilateral and let P be the point of intersection of AC and BD. Suppose that $AC + AD = BC + BD$. Prove that the internal angle bisectors of $\angle ACB, \angle ADB$ and $\angle APB$ meet at a common point.

Originally question 3 from Canadian Math Olympiad 2012.

Solved by O. Geupel; and M. Dincă and M. Miculiţa. We give the common solution of Marian Dincă and Mihai Miculiţa.

Consider the ellipse passing through A and B having the foci at C and D. The tangents to the ellipse at A respectively B meet at some point T. Denote by Q the intersection of AD and BC. We will show that the three bisectors meet at T.



To show this, it suffices to prove that T is equidistant from the lines AQ, AP, BQ respectively BP.

Since TB is tangent to the ellipse, we have $\angle TBD \equiv \angle TBQ$, which shows that TB is the bisector of $\angle QBD$. This shows that T is equidistant from BQ and BP.

In the same way, since TA is tangent to the ellipse, T is equidistant from AP and AQ.

Moreover, as T is the point of intersection of the tangents at A and B, it follows that TC and TD are the angle bisectors of BCA and BDA. This implies that T is equidistant from AP, BQ and that T is equidistant from BP, AQ.

Therefore T is equidistant from the lines AQ, AP, BQ.

OC120. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where n is a positive integer and r is a rational number. (a, b, c) is called a nice triple if a, b are positive rationals, c is a positive integer and

$$S_a(n) = (S_b(n))^c$$

for all positive integers n. Find all nice triples.

Originally question 1 from Turkey team selection test 2012, day 3.

Solved by Oliver Geupel.

It is straightforward to verify that the following triples are nice: $(q, q, 1)$ for every positive rational number q, and $(3, 1, 2)$. We show that there are no other nice triples.

Note that, for $r > 0$,

$$\frac{n^{r+1}}{r+1} = \int_0^n x^r \,\mathrm{d}x < S_r(n) < \int_0^n (x+1)^r \,\mathrm{d}x = \frac{(n+1)^{r+1}}{r+1}.$$

Suppose that (a, b, c) is nice.

Then, for every $n \in \mathbb{N}$,

$$\frac{n^{a+1}}{a+1} \cdot \frac{(b+1)^c}{(n+1)^{(b+1)c}} < \frac{S_a(n)}{(S_b(n))^c} = 1 < \frac{(n+1)^{a+1}}{a+1} \cdot \frac{(b+1)^c}{n^{(b+1)c}}.$$

We argue that $a + 1 = (b+1)c$. For, assume that $d = |a + 1 - (b+1)c| > 0$. If $a + 1 - (b+1)c > 0$, then

$$\frac{n^{a+1}}{a+1} \cdot \frac{(b+1)^c}{(n+1)^{(b+1)c}} = \frac{(b+1)^c}{a+1} \cdot \left(\frac{n}{n+1}\right)^{(b+1)c} \cdot n^d \to \infty$$

for $n \to \infty$, a contradiction.

On the other hand, if $a+1-(b+1)c<0$, then

$$\frac{(n+1)^{a+1}}{a+1}\cdot\frac{(b+1)^c}{n^{(b+1)c}}=\frac{(b+1)^c}{a+1}\cdot\left(\frac{n+1}{n}\right)^{a+1}\cdot\frac{1}{n^d}\to 0\,,$$

for $n\to\infty$, a contradiction. Consequently, $a+1=(b+1)c$.

Moreover, for every $n\in\mathbb{N}$,

$$\left(\frac{n}{n+1}\right)^{a+1}<\frac{a+1}{(b+1)^c}<\left(\frac{n+1}{n}\right)^{a+1},$$

hence $a+1=(b+1)^c$.

If $c=1$ then obviously $a=b$.

If $c=2$ then $2(b+1)=(b+1)^2$, which implies $b=1$ and $a=3$.

Finally assume $c\geq 3$. Then, the number $c=(b+1)^{c-1}$ is an integer. It follows that b itself is an integer, so that $b\geq 1$. But now $c=(b+1)^{c-1}\geq 2^{c-1}>c$, which is impossible. Consequently, our list of nice triples is complete.

Math Quotes

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks, of which the logical paradoxes constitute the most important part. For, if mathematics is to advance securely and confidently it must first set its affairs in order at home.

Kasner, E. and Newman, J., "Mathematics and the Imagination", New York: Simon and Schuster, 1940.