Ramsey's Theory Through Examples, Exercises, and Problems: Part II

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1 Introduction

In the first part of this two-part article (appearing in ***Crux***, Volume 40 (2)), we introduced, for natural numbers s and t, $s, t \geq 2$, *the Ramsey number* $R(s,t)$ as the minimum number n for which any edge 2-colouring of K_n in red and blue, contained a red K_s or a blue K_t. We also established that $R(3,3) = 6$ and that $R(2,t) = R(t,2) = t$.

In this sequel, we continue our investigation of Ramsey numbers.

2 Ramsey Numbers $R(4,3)$, $R(3,5)$, and $R(4,4)$

In Exercises 1 – 4, we will establish that $R(4,3) \leq 9$.

Suppose that the edges of the complete graph K_9 are coloured with two colours, red and blue. We need to prove that this edge 2-colouring of K_9 yields a red K_4 or a blue K_3.

Since each vertex in K_9 is incident with 8 edges we notice that there are three possible cases:

1. There is a vertex incident with at least 6 red edges.
2. There is a vertex incident with at least 4 blue edges.
3. Each vertex is incident with exactly 5 red edges and 3 blue edges.

Exercise 1 *Suppose that there is a vertex incident with at least 6 red edges. Use the fact that* $R(3,3) = 6$ *to conclude that in this case the given edge 2-colouring of* K_9 *yields a red* K_4 *or a blue* K_3*. See Figure 1.*

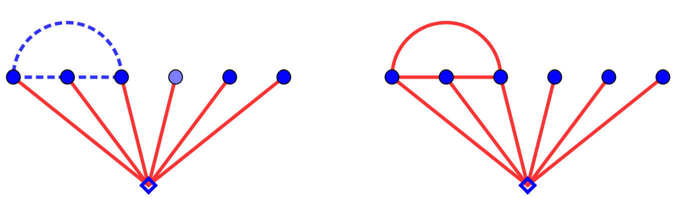

Figure 1: At least six red (solid) edges incident to the fixed vertex

Exercise 2 *Suppose that there is a vertex incident with at least 4 blue edges. Prove that in this case the given edge 2-colouring of* K_9 *yields a red* K_4 *or a blue* K_3*. See Figure 2.*

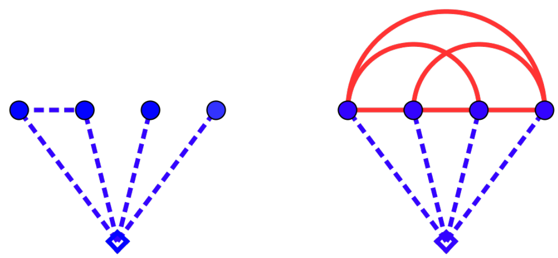

Figure 2: At least four blue (dashed) edges incident to the fixed vertex

Exercise 3 *Suppose that every vertex is incident with 5 red edges and with 3 blue edges. How many blue edges are there altogether?*

Exercise 4 *Based on Exercises 1 – 3, conclude that* $R(4,3) \leq 9$*. Justify your answer, i.e., explain what else you need to conclude that* $R(4,3) \leq 9$*.*

Problem 1 *Prove that* $R(4,3) = R(3,4) = 9$*.*

Problem 2 *Using* $R(2,5) = 5$ *and* $R(3,4) = 9$*, prove that* $R(3,5) \leq 14$*.*

Problem 3 *Prove that* $R(4,4) \leq 18$*.*

Actually it is known that $R(3,5) = 14$ and $R(4,4) = 18$. There is only one edge 2-colouring of K_{17} that avoids a monochromatic K_4!

Note that $R(4,4) = 18$ means that at any party with at least 18 people there would be either four mutual acquaintances or four mutual strangers. Interestingly enough, if one asks for the size of a party that would guarantee either five mutual acquaintances or five mutual strangers the best that we can say is that there should be no less than 43 and no more than 49 people at the party.

3 General Ramsey Numbers

We have established Ramsey numbers in a few special cases. In the rest of this note we will consider the following three questions:

Question 1 *Does the Ramsey number* $R(s,t)$ *exist for any choice of natural numbers* $s \geq 2$ *and* $t \geq 2$*?*

Question 2 *If the Ramsey number* $R(s,t)$ *exists, is its exact value known?*

Question 3 *If* $R(s,t)$ *exists and if we do not know its exact value, what are the known bounds for* $R(s,t)$*?*

We examine Question 1 through Exercises 5 – 7.

Exercise 5 *Observe that if* $s, t \in \mathbb{N}\backslash\{1\}$ *are such that*

$$s + t = 4 \text{ or } s + t = 5 \text{ or } s + t = 6$$

then $R(s, t)$ *exists.*

Exercise 6 *Suppose that* $s, t \geq 3$ *are such that* $R(s-1, t)$ *and* $R(s, t-1)$ *exist. To prove that* $R(s, t)$ *exists it is enough to prove that any 2-colouring of a complete graph* K_M *where*

$$M = R(s-1, t) + R(s, t-1)$$

yields a monochromatic K_s *or a monochromatic* K_t*. Why?*

Exercise 7 *Suppose that* $n \geq 6$ *is such that for any* $u, v \geq 3$ *such that* $u + v = n$ *the Ramsey number* $R(u, v)$ *exists. Suppose that* $s, t \geq 3$ *are such that* $s+t = n+1$*.*

1. *Conclude that* $R(s-1, t)$ *and* $R(s, t-1)$ *exist.*
2. *Prove that any 2-colouring of a complete graph* K_M *where*

   $$M = R(s-1, t) + R(s, t-1)$$

   yields a monochromatic K_s *or a monochromatic* K_t*.*
   (For a hint, see Problems 1 – 3 and Figure 3.)

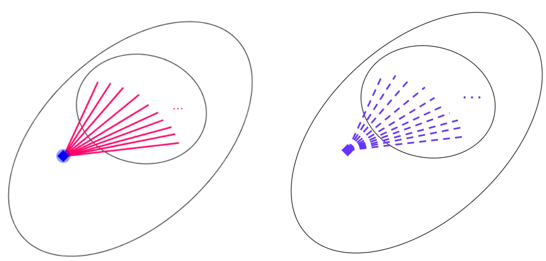

Figure 3: At least $R(s-1, t)$ red edges (left) and at least $R(s, t-1)$ blue edges (dashed, on the right).

Problem 4 *Use mathematical induction on the sum* $s+t$ *to prove that for* $s, t \geq 3$

$$R(s, t) \leq R(s-1, t) + R(s, t-1).$$

Hence, Problem 4 establishes an affirmative answer to Question 1.

It turns out that finding the exact values for $R(s, t)$ is a challenging task. In Erdős's words:

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

For an extensive account of the currently known exact values and bounds for Ramsey numbers see Section 2 in [2].

Finally, we address Question 3 through the following problem.

Problem 5 *Use mathematical induction on the sum* $s+t$ *to prove that for* $s, t \geq 2$

$$R(s,t) \leq \binom{s+t-2}{t-1}.$$

We note that Problem 5 implies Ramsey's theorem, as it was stated in Part I, in the case $r = \mu = 2$ and $n = s$.

4 Two Problems

Problem 6 *Prove that for* $s \geq 3$*,*

$$R(s,s) > 2^{s/2}.$$

(See [1].)

Problem 7 *Show that* $R(3,3,3) \leq 17$*, that is show that every 3-colouring of the edges of* K_{17} *gives a monochromatic* K_3.

References

[1] Graham, R., Rothschild, B., and Spencer, J.H., *Ramsey Theory* (2nd ed.), New York: John Wiley and Sons, 1990.

[2] Radziszowski, S.P., *Small Ramsey Numbers*, Electronic Journal of Combinatorics, Dynamic Survey DS1, revision #14 (2014), `http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS1`http://www.combinatorics.org.

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