# Excerpt from The Math Olympian

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Continued from ***Crux***, Volume 40 (2).

We left off as Bethany was working on problem #3 of The Canadian Mathematical Olympiad which asked to determine the value of:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\cdots+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

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*Calm down, Bethany, calm down. There's lots of time left. You can do this.*

I think about the soothing words of Mr. Collins, and am reminded of another important problem-solving strategy I learned from him: simplify the problem by breaking it into smaller and easier parts, in order to find a pattern.

I can do that.

I don't want to deal with the horrible expression given in the problem, a complicated sum of nearly one thousand fractions. I've seen enough contest problems to know that the number 1000 is a distracter, and that it has nothing to do with the question. By making the number big, the problem looks a lot more intimidating than it actually is.

For example, in the addition question that Mr. Collins posed to me that day, as soon as I realize that the series telescopes, it doesn't matter whether there are nine fractions or nine thousand fractions. In the former the answer is $\frac{1}{1}-\frac{1}{10}=\frac{9}{10}$ and in the latter the answer is $\frac{1}{1}-\frac{1}{9001}=\frac{9000}{9001}$. The final answer is different, but at its heart, it's the exact same problem.

I'm sure the same is true with this Olympiad problem. Especially being the first problem, I know there has to be a short and elegant solution. Remembering the advice of Mr. Collins, I decide to simplify the problem in order to discover a pattern, which will then allow me to solve the actual problem.

I change the denominator from 1000 to 4, to have just a few terms to play with. So now, instead of the exponents ranging from $\frac{1}{1000}$ to $\frac{999}{1000}$ I only have to consider $\frac{1}{4}$, $\frac{2}{4}$, and $\frac{3}{4}$.

Instead of adding 999 ugly terms as in the actual problem, I only have three terms in the simplified problem. By making the expression easier, I am hopeful that I'll discover something interesting.

So my simplified problem is to determine the value of

$$\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3}+\frac{9^{\frac{2}{4}}}{9^{\frac{2}{4}}+3}+\frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3}.$$

This looks much more reasonable. The middle expression is easy — I figured this out ten minutes earlier.

$$\frac{9^{\frac{2}{4}}}{9^{\frac{2}{4}}+3}=\frac{9^{\frac{1}{2}}}{9^{\frac{1}{2}}+3}=\frac{\sqrt{9}}{\sqrt{9}+3}=\frac{3}{3+3}=\frac{3}{6}=\frac{1}{2}.$$

As I ponder how to calculate the values of $\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3}$ and $\frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3}$, a few ideas occur to me. I scribble some calculations on my notepad, add up the two fractions, and am surprised that the sum is exactly one.

$$\frac{9^{\frac{1}{4}}}{9^{\frac{1}{4}}+3}+\frac{9^{\frac{3}{4}}}{9^{\frac{3}{4}}+3}=1.$$

Interestingly, the first and last terms of my simplified problem add up to 1. I have a hunch that this might also be true in the more complicated Olympiad problem with 999 terms.

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}}+3}+\cdots+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}.$$

To my delight, the hunch is correct.

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}}+3}+\frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}}+3}=1.$$

I run through the calculations one more time, double-checking that I haven't made any mistakes. Yes, the terms in the numerator perfectly match the terms in the denominator, and the sum is indeed one.

I wonder whether this pattern continues, and am shocked to discover that

$$\frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}}+3}+\frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}}+3}=1.$$

I suddenly feel a lump in my throat. I know how to solve the Olympiad problem.

The key insight is staring me in the face.

All I need to do is apply the technique I discovered in Mrs. Ridley's class seven years ago, when I was in Grade 5. I can't believe it.

It's the Staircase.

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"The Math Olympian" was published by FriesenPress in January 2015. For more information, please visit `www.richardhoshino.com`.