CONTEST CORNER SOLUTIONS

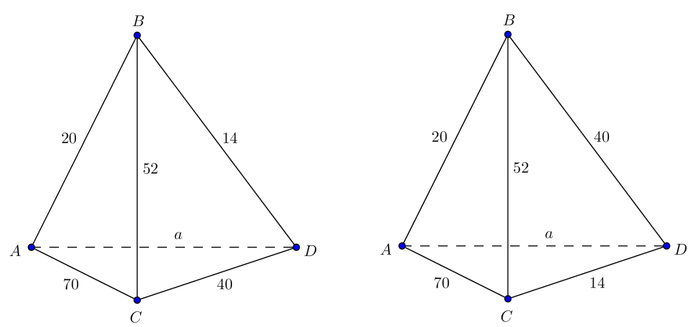
CC66. The lengths of all six edges of a tetrahedron are integers. The lengths of five of the edges are 14, 20, 40, 52, and 70. Determine the number of possible lengths for the sixth edge.

Originally problem 25 from 2002 Fermat Contest.

Solved by R. Hess ; and N. Stanciu and T. Zvonaru. We present a hybrid solution.

Let $ABCD$ be the given tetrahedron and let a be the length of the sixth edge, which we assume is AD. The edges of $\triangle ABC$ and $\triangle BCD$ must satisfy the triangle inequality, and the two triangles have exactly one edge in common. Therefore, we may assume, after a few simple calculations, that $\triangle ABC$ has edge lengths $20, 52$, and 70 and $\triangle BCD$ has edge lengths $14, 40$, and 52, and that $|BC| = 52$.

This yields two possibilities for the edge lengths of the faces, $\triangle ABC$, $\triangle BCD$, $\triangle ABD$, and $\triangle ACD$. They are either : Case 1, $(20, 52, 70)$, $(52, 40, 14)$, $(20, 14, a)$, and $(70, 40, a)$, respectively (on the left below) ; or Case 2, $(20, 52, 70)$, $(52, 14, 40)$, $(20, 40, a)$, and $(70, 14, a)$, respectively (on the right below).



The triangle inequality gives the necessary bounds on a.

Case 1. The inequalities $20 + 14 > a$, $20 + a > 14$, $14 + a > 20$, $70 + 40 > a$, $40 + a > 70$, and $70 + a > 40$ imply that $30 < a < 34$, of which integer solutions are $a \in \{31, 32, 33\}$.

Case 2. The inequalities $20 + 40 > a$, $20 + a > 40$, $40 + a > 20$, $70 + 14 > a$, $14 + a > 70$, and $70 + a > 14$ imply that $56 < a < 60$, of which integer solutions are $a \in \{57, 58, 59\}$.

So, there are exactly six possible lengths for the sixth edge : 31, 32, 33, 57, 58, 59.

CC67. The twenty volumes, clearly numbered 1 to 20, of an encyclopedia are to be arranged on a shelf. If ten volumes have blue covers, six have red covers, and the remainder have green covers, determine in how many ways the books can be arranged so that no two books of the same colour are side by side.

Originally 1978 Descartes Contest, problem 7.

Solved by Edward Wang, whose solution we present below.

First, permute the 10 blue books. This can be done in 10! ways. For convenience, call each of the empty spaces between two adjacent blue books a slot. By the given conditions, each of these 9 slots must be filled with at least one, but at most two non-blue books. Let's denote the leftmost book L and the rightmost book R. It is clear that L and R cannot both be non-blue. Considering the colours of L and R leads us to two separate cases.

Case (i). Either L or R (but not both) is occupied by a non-blue book. Clearly there are $C(6,1) \times 2 + C(4,1) \times 2 = 20$ ways of choosing and placing this book. Then the 9 remaining non-blue books can be arranged in the 9 slots in 9! ways. Hence, there are $20 \times 9!$ such arrangements.

Case (ii). Both L and R are blue. In this case we must insert two books, one red and one green, in one of the 9 slots and then exactly one non-blue book in each of the 8 remaining slots. This can be done in $C(9,1) \times C(6,1) \times C(4,1) \times 2 \times 8! = 48 \times 9!$ ways.

Combining our two cases we conclude that the total number of permissible arrangements is $10!(20 \times 9! + 48 \times 9!) = 68 \times 10! \times 9! = 89543688192000$.

CC68. A family of straight lines is determined by the condition that the sum of the reciprocals of the x and y intercepts is a constant k for each line in the family. Show that all members of the family are concurrent.

Originally 1977 Descartes Contest, problem 8.

Solved by M. Coiculescu; R. Hess; K. Zelator; and N. Stanciu and T. Zvonaru. We present a solution based on those of Konstantine Zelator and Matei Coiculescu (done independently).

The equation of any line can be written as $ax+by=1$ for some constants a, b. The x- and y-intercepts of such a line are $1/b$ and $1/a$, respectively. Then the defining condition is that $a+b=k$, so $a=k-b$, and any line in the family can be written as

$$(k-b)x + by = 1$$

for some b. Given another such line

$$(k-c)x + cy = 1, \ c \neq b,$$

we subtract the second equation from the first to find the point of intersection :

$$\begin{aligned} (c-b)x + (b-c)y &= 0 \\ (b-c)y &= (b-c)x \end{aligned}$$

and hence $x = y$. So $(k-b)x + bx = 1$, and $kx = 1$.

Thus, $x = 1/k$, provided that $k \neq 0$, which implies $y = 1/k$. Since this is independent of b and c, every line in the family passes through $(1/k, 1/k)$.

If $k = 0$, then $b = -a$, so all the lines in the family have slope 1 and are parallel.

CC69. The Fibonacci sequence is defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. A Pythagorean triangle is a right-angled triangle with integer side lengths. Prove that f_{2k+1} is the hypotenuse of a Pythagorean triangle for every positive integer k with $k \geq 2$.

Originally from 2010 Sun Life Financial Rêpechage Competition, problem 5.

Solved by Š. Arslanagić; M. Bataiile; D. E. Manes; H. Ricardo; T. Zvonaru and N. Stanciu. We present the solution by Henry Ricardo.

We will use the identity $f_{2k+1} = f_k^2 + f_{k+1}^2$. We get

$$\begin{aligned} f_{2k+1}^2 &= (f_k^2 + f_{k+1}^2)^2 \\ &= f_k^4 + f_{k+1}^4 + 2f_k^2 f_{k+1}^2 \\ &= (f_k^4 - 2f_k^2 f_{k+1}^2 + f_{k+1}^4) + 4f_k^2 f_{k+1}^2 \\ &= (f_k^2 - f_{k+1}^2)^2 + (2f_k^2 f_{k+1}^2)^2. \end{aligned}$$

Thus f_{2k+1} is the hypotenuse of a Pythagorean triangle with legs

$$f_{k+1}^2 - f_k^2 \text{ and } 2f_k f_{k+1}.$$

CC70. The game of Square Meal is played with a heap of peanuts, initially containing N nuts. The two players take it in turns to eat a positive square number $(1, 4, 9, \ldots)$ of nuts. Whoever eats the last nut wins. For which values of N can the first player always win?

Originally Question 6 from 2004 APICS Math Competition.

One incorrect solution was received. This is, in fact, an open problem.