SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3811. *Proposed by Jung In Lee.*

Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers a and b, $af(a+b) + bf(a) + b^2$ is a perfect square.

No solutions were received for this problem. The problem remains open.

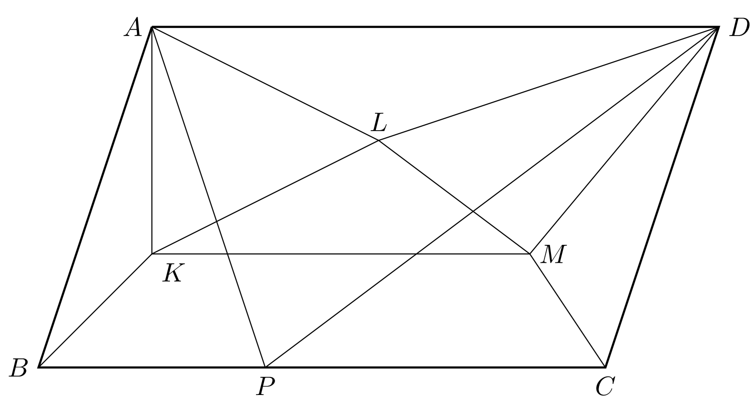
3812. *Proposed by George Apostolopoulos.*

Let $ABCD$ be a parallelogram and P be a point on side BC. Let K, L, and M be the centroids of triangles PAB, PAD and PCD, respectively. Prove that

$$[AKL] + [DLM] = [BKMC],$$

where $[\cdot]$ represents the area.

Solved by AN-anduud Problem Solving Group; M. Bataille; P. De; N. Evgenidis; O. Geupel; J. Heuver; O. Kouba; M. Modak; C. Mortici; C. Sánchez-Rubio; Skidmore College Problem Solving Group; N. Stanciu and T. Zvonaru; E. Swylan; and the proposer. We present the solution of Michel Bataille.



We use areal coordinates with reference to triangle ABC.

Let $P = tB + (1-t)C$ where $t \in [0,1]$. Observing that $D = A - B + C$, we have

$$\begin{aligned} 3K &= A + B + P = A + (1+t)B + (1-t)C \\ 3L &= A + P + D = 2A + (t-1)B + (2-t)C \\ 3M &= A + (t-1)B + (3-t)C \end{aligned}$$

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It follows that

$$\begin{aligned}[AKL] &= \frac{1}{9}|\delta_1|[ABC],\\ [DLM] &= \frac{1}{9}|\delta_2|[ABC],\\ [BKM] &= \frac{1}{9}|\delta_3|[ABC],\\ [BMC] &= \frac{1}{3}|\delta_4|[ABC],\end{aligned}$$

where $\delta_i,\ i = 1, 2, 3, 4$ are the following determinants, with columns from the areal coordinates of the vertices:

$$\delta_1 = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1+t & t-1 \\ 0 & 1-t & 2-t \end{vmatrix}, \quad \delta_2 = \begin{vmatrix} 1 & 2 & 1 \\ -1 & t-1 & t-1 \\ 1 & 2-t & 3-t \end{vmatrix},$$

$$\delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1+t & t-1 \\ 0 & 1-t & 3-t \end{vmatrix}, \quad \delta_4 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & t-1 & 0 \\ 0 & 3-t & 1 \end{vmatrix}.$$

A simple calculation gives $\delta_1 = 3 - t,\ \delta_2 = 2 + t,\ \delta_3 = 2,\ \delta_4 = 1$ and so

$$[AKL]+[DLM] = (3-t+2+t)\frac{[ABC]}{9} = \frac{5 \cdot [ABC]}{9} = [BKM]+[BMC] = [BKMC].$$

3813. *Proposed by Michel Bataille.*

Find the smallest constant C such that the inequality

$$(a^7 + b^7 + c^7)^6 \le C(a^6 + b^6 + c^6)^7$$

holds for all real numbers a, b, c such that $a + b + c = 0$.

Solved by R. Barbara; R. Hess; O. Kouba; K.-W. Lau; N. Hodzić and S. Malikić; C. Mortici; P. Perfetti; S. Wagon; and the proposer. There was one incomplete solution. We present the solution of Roy Barbara, which is both efficient and provides a generalization.

The answer is

$$\frac{(2^6-1)^6}{2(1+2^5)^7} = \frac{3^5 \cdot 7^6}{2 \cdot 11^7} = \frac{28588707}{38974342} = 0.733526\ldots$$

We prove a more general result. Consider positive integers p and q both of which are odd and m and n both of which are even that satisfy $pm = qn$. Then the smallest constant C for which the inequality

$$(a^p + b^p + c^p)^m \le C(a^n + b^n + c^n)^q$$

holds for all real numbers a, b, c with $a+b+c=0$ is

$$C_0 \doteq \frac{(2^p-2)^m}{(2^n+2)^q}.$$

If $abc = 0$ the left member is 0 and the inequality holds for all positive C. Suppose henceforth that $abc \neq 0$. Since $a+b+c=0$, two of the variables have one sign and the other the opposite. Since the expressions on both sides of the inequality do not change if we replace each variable by its negative, we may suppose wolog that $a, b > 0$ and $c < 0$. Furthermore, since both sides of the inequality are homogeneous with degree $pm = qn$, we may suppose that $a+b=2$. Therefore, the given inequality is equivalent to

$$(2^p - (a^p+b^p))^m \leq C(2^n + a^n + b^n)^q$$

with a, b positive and summing to 2.

Using the fact that $2^p - (a^p + b^p) \geq 0$ and the convexity of the function x^k for $k \geq 1$, we have that

$$a^p + b^p \geq 2\left(\frac{a+b}{2}\right)^p$$

and

$$a^n + b^n \geq 2\left(\frac{a+b}{2}\right)^n,$$

whence

$$\begin{aligned}(2^p - (a^p+b^p))^m &\leq (2^p-2)^m = C_0(2^n+2)^q \\ &\leq C_0(2^n+a^n+b^n)^q.\end{aligned}$$

Since equality occurs when $a=b=1$, we conclude that C_0 is the minimum value of C.

Editor's comments. Because of the condition $a+b+c=0$ and the homogeneity of the inequality, many solvers reduced the problem to maximizing a function of a single variable, for example $f(x) = (x^7+1-(x+1)^7)^6(x^6+1+(x+1)^6)^{-7}$. Several solvers relied on mathematical software to negotiate the technical complexities. Stan Wagon of Macalester College used `Mathematica` and Lagrange Multipliers to optimize $(a^7+b^7+c^7)^6(a^6+b^6+c^6)^{-7}$. Replacing 6 and 7 by low values of m and $m+1$, he found that the optimal values were rational and wondered whether this was always so.

Another approach was taken by the proposer and one other submitter. The numbers a, b, c are roots of the cubic

$$(x-a)(x-b)(x-c) = x^3 - qx - r,$$

where $q = -(ab+bc+ca)$ and $r = abc$. Consider the case that $r \neq 0$. For $n \geq 1$, let $s_n = a^n + b^n + c^n$. Then $s_1 = 0$, $0 < s_2 = 2q$, $s_3 = s_1^3 + 3(qs_1 + 3r) - 6r = 3r$

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and $s_n = qs_{n-2} + rs_{n-3}$ for $n \geq 4$. This leads to $s_4 = qs_2 + rs_1 = 2q^2$, $s_5 = 5qr$, $s_6 = 2q^3 + 3r^2$ and $s_7 = 7q^2r$.

The proposer expressed the inequality in terms of s_2 and s_3 and an adroit use of a weighted arithmetic-geometric means inequality led to a successful conclusion. The other submitter did not fare so well. The given inequality was rewritten as $7^6q^{12}r^6 \leq C(2q^3 + 3r^2)^7$ or

$$\frac{t^4}{(3+2t)^7} \leq \frac{C}{7^6},$$

where $t = q^3/r^2 > 0$. For $t > 0$, the left side assumes its maximum value of $16/7^7$ at $t = 2$, so that the minimum value of C for which the inequality is satisfied is apparently $16/7$.

But does t in fact range over all of the positive reals? Since $q = \frac{1}{2}(a^2+b^2+c^2) \geq \frac{3}{2}(a^2b^2c^2)^{1/3}$, we have that

$$\frac{q^3}{r^2} = \frac{1}{8} \cdot \frac{(a^2+b^2+c^2)^3}{a^2b^2c^2} \geq \frac{27}{8} > 2.$$

The restriction that $a + b + c = 0$ will push this lower bound even higher since the condition $a = b = c$ for equality cannot occur. We note that $(a, b, c) = (1, 1, -2)$ corresponds to $t = 27/4$ and the value $3^5/2 \cdot 11^7$ for $t^4(3 + 2t)^{-7}$.

3814. *Proposed by Marcel Chiriţă.*

Prove that for any number x in the closed interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M in the plane of the square $ABCD$ such that

$$x = \frac{AM + MC}{BM + MD}.$$

Solved by A. Alt; AN-anduud Problem Solving Group; G. Apostolopoulos; D. Bailey, E. Campbell, and C. Diminnie; R. Barbara; M. Bataille, P. De; O. Geupel; J. Hawkins and D. Stone; O. Kouba; S. Malikić; C. Mortici; C. Sánchez-Rubio; E. Swylan; D. Văcaru; and the proposer. We present 2 solutions.

Solution 1, by Prithwijit De.

Because we deal with ratios of segments, we can place the square in the Cartesian plane so that the coordinates of A, B, C and D are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ respectively. The coordinates of any point M in the line segment BD may be assumed to be $(0, m)$, for some real number $m \in [-1, 1]$. For such a point M, $BM + MD = 2$ and $AM + MC = 2\sqrt{1 + m^2}$. Therefore

$$\frac{AM + MC}{BM + MD} = \sqrt{1 + m^2}.$$

Observe that as M moves along the line segment BD, $\sqrt{1 + m^2}$ decreases continuously from $\sqrt{2}$ to 1, then returns to $\sqrt{2}$, thereby covering the interval $[1, \sqrt{2}]$. If

M is taken on the line segment AC then we may assume that its coordinates are $(m, 0)$ for some real number $m \in [-1, 1]$. For such a point M,

$$\frac{AM + MC}{BM + MD} = \frac{1}{\sqrt{1 + m^2}}.$$

Observe that as M moves along the line segment AC, $\frac{1}{\sqrt{1+m^2}}$ increases continuously from $\frac{\sqrt{2}}{2}$ to 1 and then returns to $\frac{\sqrt{2}}{2}$, thereby covering the interval $[\frac{\sqrt{2}}{2}, 1]$. Thus for any x in the interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M on a diagonal of the square such that

$$x = \frac{AM + MC}{BM + MD}.$$

Solution 2 is a composite of similar arguments from George Apostolopoulos, Omran Kouba and Marcel Chiriţă.

Let M be any point in the plane of the square $ABCD$, and consider the function

$$f(M) = \frac{AM + MC}{BM + MD}.$$

Since its denominator does not vanish, f is a continuous function. Moreover, $f(A) = \sqrt{2}/2$ and $f(B) = \sqrt{2}$. Thus, by the intermediate value theorem we know that for any number x in the closed interval $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$, there exists a point M such that $f(M) = x$, which is the desired conclusion.

But we can say more, namely, that the equation $f(M) = x$ has a solution M in the plane if and only if x belongs to $\left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$. Indeed, note that for nonnegative u and v we have

$$u^2 + v^2 \le (u + v)^2 \le 2(u^2 + v^2),$$

with equality on the left when $uv = 0$, and on the right when $u = v$. So, if a, b, c and d are nonnegative real numbers such that $a^2 + b^2 = c^2 + d^2$ then

$$\frac{1}{\sqrt{2}} \le \frac{a + b}{c + d} \le \sqrt{2}. \qquad (1)$$

Now, with O at the centre of the square we have $\overrightarrow{OA} = -\overrightarrow{OC}$ and $\overrightarrow{OB} = -\overrightarrow{OD}$, so that for every point M,

$$\begin{aligned} AM^2 + CM^2 &= (\overrightarrow{OM} - \overrightarrow{OA})^2 + (\overrightarrow{OM} + \overrightarrow{OA})^2 = 2OM^2 + 2OA^2 \\ BM^2 + DM^2 &= (\overrightarrow{OM} - \overrightarrow{OB})^2 + (\overrightarrow{OM} + \overrightarrow{OB})^2 = 2OM^2 + 2OB^2. \end{aligned}$$

That is $AM^2 + CM^2 = BM^2 + DM^2$. (Alternatively, note that MO is the median of both triangles MAC and MBD whose sides opposite M have equal lengths.) Applying (1) to $a = MA$, $b = MB$, $c = MC$ and $d = MD$, we conclude that $f(M) \in \left[\frac{\sqrt{2}}{2}, \sqrt{2}\right]$ for every point M in the plane. Moreover, analyzing the

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cases of equality in (1), we see that $f(M) = \sqrt{2}$ if and only if $M \in \{B, D\}$ and $f(M) = \sqrt{2}/2$ if and only if $M \in \{A, C\}$.

3815. *Proposed by Paolo Perfetti.*

Show that $x^x \le x^2 - x + 1$ for all $0 \le x \le 1$.

Solved by AN-anduud Problem Solving Group; Š. Arslanagić; M. Bataille; N. Evgenidis; M. Dincă; O. Furdui; O. Geupel; H. Wang and J. Woydylo; O. Kouba; K.W. Lau; A. Li; S. Malikić (2 solutions); P. McCartney; D. Smith. There were 4 solutions via numerical methods to prove claims about some inequalities, which (while correct) require approximating roots of non-algebraic equations via computer assistance. We present 4 solutions, each using more powerful theorems than the previous.

Foreword: We take $0^0 = \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1$, so we have $1 \le 1^2 - 1 + 1 = 1$, and so the following solutions will only prove the inequality for all $x \in (0, 1]$.

Solution 1, by Omran Kouba.

Since $1 - x + x^2 > 0$ for $x \in [0, 1]$ we may consider the function

$$f : (0, 1] \to \mathbb{R},\ f(x) = \ln(1 - x + x^2) - x\ln x.$$

We have

$$f'(x) = \frac{2x - 1}{1 - x + x^2} - 1 - \ln x$$

and

$$f''(x) = \frac{1 + 2x - 2x^2}{(1 - x + x^2)^2} - \frac{1}{x} = \frac{1 - x}{x(1 - x + x^2)^2} g(x)$$

where $g(x) = x^3 + x^2 + 2x - 1$. Now, g is increasing on $[0, 1]$ and $g(0)g(1) < 0$, so there is unique $x_0 \in (0, 1)$ such that $g(x_0) = 0$. Moreover, $g(x) < 0$ for $0 < x < x_0$, $g(x) > 0$ for $x_0 < x < 1$. This proves that f' is decreasing on $(0, x_0]$ and increasing on $[x_0, 1]$. From $f'(1) = 0$ and $\lim_{x\to 0^+} f'(x) = +\infty$ we conclude that there is a unique $x_1 \in (0, x_0)$ with $f'(x_1) = 0$ and that $f'(x)$ has the sign of $x_1 - x$ on the interval $(0, 1]$. Therefore, f is increasing on $(0, x_1]$ and decreasing on $[x_1, 1]$. But $\lim_{x\to 0^+} f(x) = 0$, and $f(1) = 0$. The next table of variations illustrates this discussion:

x	0		x_1		x_0		1
$g(x)$		$-$			0	$+$	
$f'(x)$	$+\infty$	$\searrow$	0	$\searrow$	$\smile$	$\nearrow$	0
$f'(x)$		$+$	0	$-$		$-$	
$f(x)$	0	$\nearrow$	$\frown$		$\searrow$		0

So $f(x) > 0$ for $x \in (0, 1)$, and the proposed inequality follows.

Solution 2, by Michel Bataille.

Equality holds when $x = 1$. From now on, we suppose that $x \in (0,1)$. Let $h = 1 - x$. Then $h \in (0,1)$ and the required inequality becomes

$$e^{(1-h)\ln(1-h)} \leq 1 - h + h^2$$

or, equivalently,

$$(1-h)\ln(1-h) \leq \ln(1 - h(1-h)) \qquad (1).$$

Since $h(1-h) \in (0,1)$ as well, (1) rewrites as

$$-(1-h)\sum_{n=1}^{\infty} \frac{h^n}{n} \leq -\sum_{n=1}^{\infty} \frac{h^n(1-h)^n}{n}$$

that is,

$$\sum_{n=1}^{\infty} \frac{h^n}{n}\left((1-h) - (1-h)^n\right) \geq 0.$$

But this inequality clearly holds since $h > 0$ and for all positive integer n,

$$(1-h) - (1-h)^n = (1-h)(1 - (1-h)^{n-1}) \geq 0$$

(recalling that $0 < 1 - h < 1$). The proof is complete.

Solution 3, by Salem Malikić.

Make the substitution $a = \frac{1}{x}$, so $a \geq 1$. Then the required inequality is equivalent to

$$\left(\frac{1}{a}\right)^{\frac{1}{a}} \leq \frac{1}{a^2} - \frac{1}{a} + 1,$$

which, after rearranging and raising both sides to the power of a, becomes

$$\frac{1}{a} \leq \left(1 + \frac{1-a}{a^2}\right)^a.$$

We now use Bernoulli's inequality in the form $1 + bx \leq (1+x)^b$ for $x \geq -1$ and $b \geq 1$:

$$\frac{1}{a} = 1 + a \cdot \frac{1-a}{a^2} \leq \left(1 + \frac{1-a}{a^2}\right)^a,$$

so we are done.

Solution 4, by both the AN-anduud Problem Solving Group and Nikolaos Evgenidis.

By the weighted AM-GM inequality, we have, for $x \in (0,1]$:

$$x^2 - x + 1 = x \cdot x + (1-x) \cdot 1 \geq x^x \cdot 1^{1-x} = x^x.$$

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Editor's comments. A variation of Solution 2 involved the use of the binomial expansion, by H. Wang and J. Woydylo. There were variations on Solution 3, usually a different substitution before applying Bernoulli's inequality. V. Konećny commented that this is one part of the two-sided inequality

$$\frac{1}{2-x} < x^x < x^2 - x + 1,$$

for $x \in (0,1)$, which with proof via the weighted AM-GM inequality is found in a text by Jiří Herman of Masaryhova University, where Konećny studied 60 years ago.

3816. *Proposed by Mehmet Şahin.*

Let ABC be a right triangle with right angle at C, and let D be the foot of the altitude from C. Let I_1 and I_2 be the incentres of triangles CAD and CBD, respectively. Let ρ and r be the inradii of triangles I_1DI_2 and ABC, respectively. Prove that

$$\frac{\rho}{r} \le \frac{1}{2+\sqrt{2}}.$$

Solved by M. Amengual Covas; AN-anduud Problem Solving Group; G. Apostolopoulos; Š. Arslanagić; M. Bataille; P. De; N. Evgenidis; O. Geupel; O. Kouba; K. W. Lau; S. Malikić; M. R. Modak; N. Stanciu and T. Zvonaru; C. Sánchez-Rubio; E. Swylan; D. Văcaru; and the proposer. We present the solution by Omran Kouba.

Let us denote BC, CA and AB by a, b and c respectively. Also, let r_1 and r_2 be the inradii of triangles CAD and CBD, respectively.

First we note that $\angle I_1DI_2 = 90^\circ$ since I_1D and I_2D are the internal and external bisectors of $\angle ADC$. From the similarity of triangles CAD and BAC we conclude that $\frac{I_1D}{IC} = \frac{CA}{AB}$; analogously, from the similarity of triangles CBD and ABC we conclude that $\frac{I_2D}{IC} = \frac{BC}{AB}$. Thus,

$$\frac{I_1D}{I_2D} = \frac{AC}{AB}.$$

This proves that the right triangles I_1DI_2 and ACB are similar and consequently

$$\frac{\rho}{r} = \frac{I_1D}{AC} = \frac{\sqrt{2}\,r_1}{AC}.$$

But from the similarity of triangles CAD and BAC we see that $\frac{r_1}{AC} = \frac{r}{AB}$. So, we have proved that

$$\frac{\rho}{r} = \sqrt{2}\cdot\frac{r}{c}.$$

On the other hand,

$$\frac{ab}{a+b+c} = \frac{ab(a+b-c)}{(a+b)^2-c^2} = \frac{a+b-c}{2} \le \frac{\sqrt{2(a^2+b^2)}-c}{2}$$
$$= \frac{(\sqrt{2}-1)c}{2} = \frac{c}{2(\sqrt{2}+1)} \qquad (2)$$

and the desired inequality follows by combining (1) and (2).

3817. *Proposed by Tiagran Hakobyan.*

Let $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. Let $p_1 < p_2 < p_3 < \dots$ be the set of primes in the progression $\{ak + b\}_{k=0}^{\infty}$. Consider

$$\alpha = 0.p_1p_2p_3\cdots,$$

where the digits of the prime numbers $p_1, p_2, p_3, \dots$ placed side by side form the digits of α. Prove that α is irrational.

Solved by R. Barbara; O. Guepel; and the proposer. There was one incomplete solution. We present 2 solutions.

Solution 1, by the proposer.

By Dirichlet's theorem, there are infinitely many primes in the progression

$$\{ak + b\}_{k=0}^{\infty},$$

so the decimal expansion of α cannot terminate. Suppose that α is rational. Then its decimal expansion is eventually periodic with some positive period of length u consisting of digits not all zero. Pick a prime $q = ak_0 + b$ exceeding 5 in the sequence; suppose that q has v digits. Then, for $k \ge k_0$,

$$ak + b = a(k - k_0) + q,$$

so that the digits of the primes of the form $an + q$ with $n \ge 0$ constitute the tail of the decimal expansion of α.

Since $\gcd(a \cdot 10^m, q) = 1$, the subsequence $\{a \cdot 10^m \cdot k + q\}$ with $m = 2u + v$ also has infinitely many primes, and each such prime will be responsible for a block of at least $2u$ zeros in the expansion of α. Since any succession of $2u$ digits contains one full period, we are led to a contradiction.

Solution 2, by Oliver Geupel.

Assume that α is eventually periodic with period length u. By omitting a suitable finite number of initial terms in the sequence $\{ak + b\}$, we may assume that α is actually periodic. For $i \ge 1$, let n_i be the number of primes in the sequence with i decimal digits. The the sum of the reciprocals of primes p_j with i digits satisfies

$$\sum \left\{ \frac{1}{p_j} : 1 + \sum_{t=1}^{i-1} n_t \le j \le \sum_{t=1}^{i} n_t \right\} \le \frac{n_i}{10^{i-1}},$$

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since each p_j exceeds 10^{i-1}.

Suppose that $n_i > u$. Then there is an increasing succession of $u+1$ primes each with i digits that give rise to $(u+1)i = ui + i$ consecutive digits of α. Because ui is a multiple of u, the first i of these digits is equal to the last i, so that two of the primes are equal and we get a contradiction. Thus, $n_i \leq u_i$ for all i and the sum of the reciprocals of the primes in the sequence does not exceed

$$u\sum_{i=1}^{\infty}(1/10^{i-1}) = 10u/9.$$

But this contradicts the strong form of Dirichlet's theorem, so the sum diverges.

Editor's comments. Barbara observed that this result holds for any base of numeration. Geupel mentioned that he was inspired by the paper B. Sung, *When is a decimal expansion irrational?* Resonance (Indian Academy of Science) 9 (2004), 78-80.

3818. *Proposed by José Luis Díaz-Barrero.*

Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{(\sqrt{a}+\sqrt{b})^4}{a+b} + \frac{(\sqrt{b}+\sqrt{c})^4}{b+c} + \frac{(\sqrt{c}+\sqrt{a})^4}{c+a} \geq 24.$$

Solved by AN-anduud Problem Solving Group; G. Apostopoulos; Š. Arslanagić; M. Bataille; E. Campbell, D.T. Bailey and C. Diminnie; P. De; M. Dincă; N. Evgenidis; K. W. Lau; S. Malikić; P. McCartney; M. Modak; C. Mortici; D. Smith; D. Văcaru; S. Wagon; H. Wang and J. Wojdylo; and the proposer. We present 2 solutions.

Solution 1, provided by most of the solvers.

From the inequality $(x+y)^4 - 8xy(x^2+y^2) = (x-y)^4 \geq 0$, we deduce that the left side of the inequality is not less than $8(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$. An application of the arithmetic-geometric means inequality yields the desired result.

Solution 2, by Nikolaos Evgenidis.

Let $(x,y,z) = (\sqrt{a}, \sqrt{b}, \sqrt{c})$. Applying the Cauchy-Schwarz Inequality $(\sum u_i v_i)^2 \leq (\sum u_i^2)(\sum v_i^2)$ with $u_1 = (x+y)^2/\sqrt{x^2+y^2}, v_1 = \sqrt{x^2+y^2}$, we see that the left side of the inequality is not less than

$$\frac{[(x+y)^2+(y+z)^2+(z+x)^2]^2}{2(x^2+y^2+z^2)}.$$

Since $xy + yz + xx \geq 3(xyz)^{1/3} = 3$ and

$$\begin{aligned}[(x+y)^2 + (y+z)^2 + (z+x)^2]^2 &= 4(x^2+y^2+z^2+xy+yz+zx)^2 \\ &\geq 4(x^2+y^2+z^2+3)^2 \\ &= 4[(x^2+y^2+z^2-3)^2 + 12(x^2+y^2+z^2)] \\ &\geq 48(x^2+y^2+z^2),\end{aligned}$$

the desired inequality follows.

3819. *Proposed by Francisco Javier Garciá Capitán.*

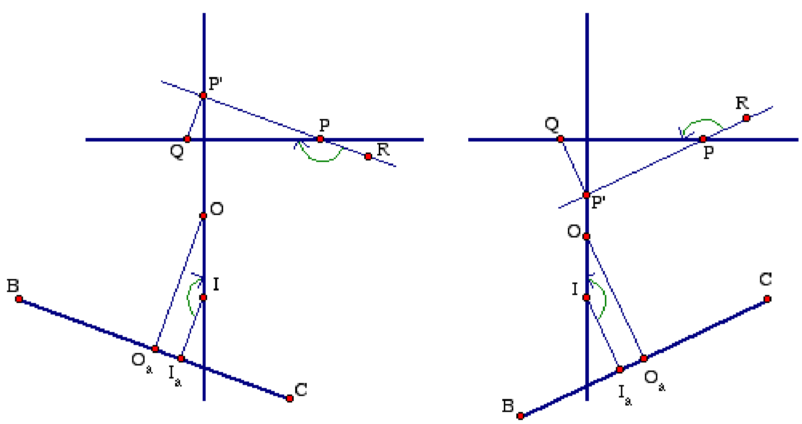
Let ABC be a triangle with circumcentre O and incentre I. Let ℓ be any line that is perpendicular to OI. Prove that for any point P on ℓ that is inside the triangle, the sum of the distances from P to the sides of ABC is constant.

Solved by M. Bataille; O. Kouba; E. Swylan; T. Zvonaru and N. Stanciu; and the proposer. We present the solution by Edmund Swylan, with details added by the editor.

With the use of signed distances, there is no need to restrict P to the inside of the given triangle. Specifically, we define the distance $d(P, YZ)$ of the point P to the side YZ of a triangle XYZ to be positive if and only if P and X lie in the same half-plane defined by the line YZ.

Here, we are given two points P and Q in the plane of an arbitrary nonequilateral triangle ABC, so that $PQ \perp OI$, and we label the points so that a rotation of $90°$ about P takes the vector $\overrightarrow{IO}$ into a vector that points in the same direction as $\overrightarrow{PQ}$. We are to prove that

$$\Sigma := (d(Q,BC) - d(P,BC)) + (d(Q,CA) - d(P,CA)) + (d(Q,AB) - d(P,AB)) = 0.$$



Define I_a and O_a to be the feet of the perpendiculars to BC from I and O, respectively, and $\theta_a = \angle I_a IO$ to be a signed angle (positive if labeled counterclockwise).

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We introduce the notation $\overline{XY}$ for the signed distance from X to Y, where we take $\overline{IO}, \overline{AB}, \overline{BC}$, and $\overline{CA}$ to define the positive direction of the lines they determine. In this way,

$$\overline{I_aO_a} = \overline{IO}\sin\theta_a \tag{1}$$

regardless of how the lines IO and BC are related, as depicted in the figure where both $\overline{I_aO_a}$ and θ_a are negative on the left, and both are positive on the right.

Let P' be the point where the parallel to BC through P meets the perpendicular to BC through Q. We define the sign of $\overline{P'Q}$ so that $\overline{P'Q} = d(Q, BC) - d(P, BC)$. Let R be any point on the line $P'P$ for which the vectors $\overrightarrow{BC}$ and $\overrightarrow{PR}$ point in the same direction. Then because a rotation through 90° about P takes the vectors $\overrightarrow{II_a}$ and $\overrightarrow{IO}$ into vectors that point in the same direction as $\overrightarrow{PR}$ and $\overrightarrow{PQ}$, respectively, we have $\theta_a = \angle RPQ$. Moreover, $\angle RPQ$ has the same sign as $\overline{P'Q}$ regardless of how the lines IO and BC are related, as depicted in the figure where both $\overline{P'Q}$ and $\angle RPQ$ are negative on the left, and both are positive on the right. We conclude that

$$d(Q, BC) - d(P, BC) \;=\; \overline{P'Q} \;=\; \overline{PQ}\sin\theta_a,$$

with analogous expressions for $d(Q, CA) - d(P, CA)$ and $d(Q, AB) - d(P, AB)$. Thus, with I_b, O_b, I_c, and O_c the respective projections of I and O on the sides CA and AB, and $\theta_b := \angle I_bIO, \theta_c := \angle I_cIO$, we have reduced our sum to

$$\Sigma = \overline{PQ}(\sin\theta_a + \sin\theta_b + \sin\theta_c).$$

We now compare Σ to the expression we get by adding together the equalities analogous to (1), namely

$$\overline{I_aO_a} + \overline{I_bO_b} + \overline{I_cO_c} = \overline{IO}(\sin\theta_a + \sin\theta_b + \sin\theta_c). \tag{2}$$

Because we assume that neither $\overline{IO}$ nor $\overline{PQ}$ can be zero, Σ can vanish if and only if the sum of the three sines is zero, which can happen if and only if the sum in (2) is zero. But

$$\begin{aligned}\overline{I_aO_a} + \overline{I_bO_b} + \overline{I_cO_c} &= (\overline{BO_a} - \overline{BI_a}) + (\overline{CO_b} - \overline{CI_b}) + (\overline{AO_c} - \overline{AI_c}) \\ &= (\overline{BO_a} + \overline{CO_b} + \overline{AO_c}) - (\overline{BI_a} + \overline{CI_b} + \overline{AI_c}).\end{aligned}$$

Because each of the sums in the last line is equal to the semiperimeter of the triangle, their difference is zero, as desired.

Editor's comments. By coincidence, an article [2] appeared in the latest *Mathematics Magazine* that deals with issues related to our problem, namely Viviani's theorem and its extension to the result,

The sum of the distances from a point inside a triangle to the three sides takes every value from the smallest altitude of the triangle to the largest altitude.

(Viviani's theorem deals with the equilateral triangle, where the sum of the three distances equals the altitude.) Polster provides a simple pictorial proof, but the result is also an easy consequence of our solution to 3819. He refers to the article [1], where Abboud uses linear programming to prove that any triangle can be divided into parallel segments on which the sum is constant, but neither author observes that these parallel segments happen to be perpendicular to OI.

References:

[1] Elias Abboud, Viviani's theorem and its extensions, *College Math. J.* **41**:3 (May 2010) 203-211.

[2] Burkard Polster, Viviani á la Kawasaki: Take Two, *Math. Mag.* **87**:4 (October 2014) 280-283.

3820. *Proposed by Michel Bataille.*

Prove that

$$\frac{2x}{\sinh(2\tanh x)} < (\cosh x)^2 < \frac{2x}{\sinh(2\tanh x)} + x\sinh(2x)$$

for all nonzero real x.

Solved by O. Kouba; and the proposer. There was one incomplete solution, and one solution consisting solely of a Mathematica verification. We present Omran Kouba's solution.

For $t \in (0,1)$ we have

$$\frac{1-t^2}{2t}\ln\left(\frac{1+t}{1-t}\right) = \frac{(1-t^2)}{t}\tanh^{-1}(t) = (1-t^2)\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1} = 1-2\sum_{n=1}^{\infty}\frac{t^{2n}}{4n^2-1} < 1$$

and

$$\frac{\sinh(2t)}{2t} = 1 + \sum_{n=1}^{\infty}\frac{2^{2n}t^{2n}}{(2n+1)!} > 1.$$

Combining the above, we conclude that for $t \in (0,1)$ we have

$$\frac{\sinh(2t)}{2t} > (1-t^2)\frac{\tanh^{-1}(t)}{t}$$

or equivalently

$$\frac{1}{1-t^2} > \frac{2\tanh^{-1}(t)}{\sinh(2t)}.$$

Applying this with $t = \tanh x$, and noting that both sides of the obtained inequality are even functions, we obtain the first inequality.

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In a similar way, for $t \in (0,1)$ we have

$$\begin{aligned}\big(1-t^2+t\sinh(2t)\big)\frac{\tanh^{-1}(t)}{t}-\frac{\sinh(2t)}{2t}\\ &=\left(1+t^2+\sum_{n=2}^{\infty}\frac{2^{2n-1}t^{2n}}{(2n-1)!}\right)\left(\sum_{n=0}^{\infty}\frac{t^{2n}}{2n+1}\right)-\sum_{n=0}^{\infty}\frac{2^{2n}t^{2n}}{(2n+1)!}\\ &=\frac{2}{3}t^2+\sum_{n=2}^{\infty}a_nt^{2n}\end{aligned}$$

with

$$a_n=\frac{4n}{4n^2-1}+\frac{(2n^2+n-1)2^{2n}}{(2n+1)!}+\sum_{k=1}^{n-2}\frac{2^{2n-2k-1}}{(2k+1)(2n-2k-1)!}>0$$

This proves that for $t \in (0,1)$ we have $\big(1-t^2+t\sinh(2t)\big)\dfrac{\tanh^{-1}(t)}{t}>\dfrac{\sinh(2t)}{2t}$. Multiplying both sides by the positive quantity $\dfrac{2t}{(1-t^2)\sinh(2t)}$, we obtain

$$\left(\frac{2}{\sinh(2t)}+\frac{2t}{1-t^2}\right)\tanh^{-1}(t)>\frac{1}{1-t^2}.$$

Applying this, with $t=\tanh x$, and noting that both sides of the obtained inequality are even functions, the second inequality follows.

Editor's comments. The featured solution utilises a substitution to simplify the problem, and then uses power series to verify each side of the inequality, thanks to how related the individual series are. Interestingly, the left-hand inequality is provable without passing to the useful but tedious power series. S. Malikić utilised the inequality $\frac{\sinh(t)}{t}>1$ for all $t \neq 0$, as follows:

$$\begin{aligned}\frac{2x}{\sinh(2\tanh(x))}&=\frac{2\tanh(x)}{\sinh(2\tanh(x))}\frac{x}{\tanh(x)}\\ &<\frac{x}{\tanh(x)}\\ &=\frac{x}{\sinh(x)}\cosh(x)<\cosh(x)<\cosh(x)^2.\end{aligned}$$

The proposer, M. Bataille, took an even more elementary approach, by computing that the derivative of $f(x)=\cosh^2(x)\sinh(2\tanh(x))-2x$ is positive for positive x, and that $f(0)=0$, therefore proving the left-hand inequality. The right-hand inequality has seen no such simple proof, although the proposer found a simpler inequality which implies the right-hand inequality, which yields a less troublesome series computation.