THE OLYMPIAD CORNER

No. 320

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Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. Veuillez s'il vous plaît àcheminer vos soumissions à `crux-olympiad@cms.math.ca` *ou par la poste à l'adresse figurant à l'endos de la page couverture arrière. Les soumissions électroniques sont généralement préférées.*

Comment soumettre une solution. *Nous demandons aux lecteurs de présenter chaque solution dans un fichier distinct. Il est recommandé de nommer les fichiers de la manière suivante :* Nom de famille_Prénom_Numéro du problème *(exemple : Tremblay_Julie_1234.tex). De préférence, les lecteurs enverront un fichier au format LaTeX et un fichier pdf pour chaque solution, bien que les autres formats soient aussi acceptés. Nous acceptons aussi les contributions par la poste. Le nom de la personne qui propose une solution doit figurer avec chaque solution, de même que l'établissement qu'elle fréquente, sa ville et son pays ; chaque solution doit également commencer sur une nouvelle page.*

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au rédacteur au plus tard le **1er juin 2015** *; toutefois, les solutions reçues après cette date seront aussi examinées jusqu'au moment de la publication.*

Chaque problème est présenté en anglais et en français, les deux langues officielles du Canada. Dans les numéros 1, 3, 5, 7 et 9, l'anglais précédera le français, et dans les numéros 2, 4, 6, 8 et 10, le franais précédera l'anglais. Dans la section Solutions, le problème sera écrit dans la langue de la première solution présentée.

La rédaction souhaite remercier d'avoir traduit les problèmes.

OC166. Soit $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Déterminer la valeur maximale de

$$\sum_{n=1}^{10} (na_n^2 - n^2 a_n)\,.$$

OC167. Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$, telles que

$$(x-2)f(y) + f(y + 2f(x)) = f(x + yf(x))$$

pour tout $x, y \in \mathbb{R}$.

OC168. Soit $ABCD$ un carré. Déterminer tous les points P dans le plan, différents de A, B, C, D, tels que

$$\angle APB + \angle CPD = 180^\circ\,.$$

OC169. Déterminer tous les entiers positifs $n \geq 2$ tels que, pour tous les entiers $0 \leq i, j \leq n$, les nombres $i+j$ et $\binom{n}{i} + \binom{n}{j}$ ont la même parité.

OC170. Soit ABC un triangle. Les bissectrices des angles $\angle CAB$ et $\angle ABC$ intersectent les segments BC et AC à D et E respectivement. Démontrer que

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA)\,.$$

. . . .

OC166. Let $\{a_1, a_2, \cdots, a_{10}\} = \{1, 2, \cdots, 10\}$. Find the maximum value of

$$\sum_{n=1}^{10} (na_n^2 - n^2 a_n)\,.$$

OC167. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-2)f(y) + f(y + 2f(x)) = f(x + yf(x))$$

for all $x, y \in \mathbb{R}$.

OC168. Let $ABCD$ be a square. Find the locus of points P in the plane, different from A, B, C, D such that

$$\angle APB + \angle CPD = 180^\circ\,.$$

OC169. Find all positive integers $n \geq 2$ such that for all integers $0 \leq i, j \leq n$ the numbers $i + j$ and $\binom{n}{i} + \binom{n}{j}$ have the same parity.

OC170. Let ABC be a triangle. The internal bisectors of angles $\angle CAB$ and $\angle ABC$ intersect segments BC, respectively AC at D, respectively E. Prove that

$$DE \leq (3 - 2\sqrt{2})(AB + BC + CA)\,.$$

OLYMPIAD SOLUTIONS

OC106. Find all the positive integers n for which all the n digit integers containing $n-1$ ones and 1 seven are prime.

Originally question 3 from Macedonia National Olympiad 2011.

Solved by R. Hess; D. E. Manes; and D. Văcaru. We give two solutions.

Solution 1, by David E. Manes.

If $n = 1$ then the number is 7 which is prime.

If $n = 2$ then the only two possibilities are 71 or 17, which are both primes.

We claim that there is no $n \geq 3$ which works. To see this, we look at the remainder of n when divided by 6.

If $n = 6k$ or $n = 6k + 3$, the sum of the digits of all these numbers is $n + 6$ which is divisible by 3. Therefore, none of the numbers is prime.

If $n = 6k+1$ with $k \geq 1$, then since $111111 = 7*11*13$, it follows that the number

$$7\underbrace{111\ldots1}_{6k}$$

is divisible by 7, and strictly larger than 7. Therefore, it is not prime.

If $n = 6k+2$, then, as $n > 2$ it follows that $k \geq 1$. Then $n = 6k'+8$ where $k' \geq 0$. Then, as 11171111 is a multiple of 7, it follows that

$$11171111\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

If $n = 6k + 4$, then since $7111 = 13 * 547$, it follows that

$$7111\underbrace{111\ldots1}_{6k}$$

is divisible by 13, hence not prime.

Finally, if $n = 6k + 5$, then since 11711 is a multiple of 7, it follows that

$$11711\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

Solution 2, by Richard I. Hess.

It is easy to see that $n = 1, n = 2$ work.

Case $n = 3$ doesn't work because all numbers are divisible by 9.

Case $n = 4$ doesn't work because $1711 = 29 * 59$.

Case $n = 5$ doesn't work because $11711 = 7 * 1763$.

We now prove that no $n \geq 6$ works. First, let us observe that for each n every number has the form

$$\underbrace{111\ldots1}_{n} + 6 * 10^k ,$$

for some $1 \leq k \leq n$. Next, note that

$$7 | \underbrace{111\ldots1}_{n} \text{ if and only if } 7 | \underbrace{999\ldots999}_{n} \text{ if and only if } 7 | 10^n - 1.$$

This is equivalent to $10^n \equiv 1 \pmod 7 \Leftrightarrow 6 = \text{ord}_7(10) | n$. We split now the problem in two cases.

Case 1: $6|n$. Here all the numbers are divisible by 3, and hence none is prime.

Case 2: $6 \nmid n$. In this case, we saw above that $7 \nmid \underbrace{111\ldots1}_{n}$. Therefore, $\underbrace{111\ldots1}_{n}$ is invertible modulo 7. As 10 is a primitive root modulo 7, there exists a $1 \leq k \leq 6 \leq n$ such that

$$\underbrace{111\ldots1}_{n} \equiv 10^k \pmod 7 ,$$

Therefore, for this k, 7 divides $\underbrace{111\ldots1}_{n} + 6 * 10^k$ and hence this number is not prime.

It follows that $n = 1$ or $n = 2$.

OC107. ABC is a triangle of perimeter 4. Point X is marked on the ray AB and point Y is marked on the ray AC such that $AX = AY = 1$. BC intersects XY at M. Prove that one of the triangles ABM or ACM has perimeter 2.

Originally question 4 from Russia National Olympiad 2012, Grade 10 Day 1.

Solved by Michel Bataille whose solution we present below.

Let $BC = a, CA = b, AB = c$. From the hypotheses, $a = 4 - b - c$ and

$$\overrightarrow{AX} = \frac{1}{c}\overrightarrow{AB}, \quad \overrightarrow{AY} = \frac{1}{b}\overrightarrow{AC} .$$

In real coordinates relative to (A, B, C), we have

$$X = (c - 1 : 1 : 0), \quad Y = (b - 1 : 0 : 1), \quad XY : x + y(1 - c) + z(1 - b) = 0$$

so that $M = (0 : 1 - b : c - 1)$. Since M is interior to segment BC, $1 - b$ and $c - 1$ must have the same sign.

Without loss of generality, we suppose that $c > 1$ and $b < 1$ in what follows. Then

$$(c-b)\overrightarrow{BM} = (c-1)\overrightarrow{BC}$$

and

$$(c-b)\overrightarrow{CM} = (1-b)\overrightarrow{CB},$$

so that

$$(c-b)BM = (c-1)a = (c-1)(4-b-c), \tag{6}$$
$$(c-b)CM = (1-b)a = (1-b)(4-b-c)\,. \tag{7}$$

Also,

$$(c-b)\overrightarrow{AM} = (1-b)\overrightarrow{AB} + (c-1)\overrightarrow{AC}$$

so that

$$(c-b)^2AM^2 = (1-b)^2c^2 + (c-1)^2b^2 + (1-b)(c-1)(2\overrightarrow{AB}\cdot\overrightarrow{AC})$$

with

$$2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2+c^2-a^2 = b^2+c^2-(4-b-c)^2 = 8b+8c-2bc-16\,.$$

A short calculation gives $(c-b)^2AM^2 = (3b+3c-2bc-4)^2$, hence

$$(c-b)AM = |3b+3c-2bc-4|\,. \tag{8}$$

Now, if $3b+3c-2bc-4 \geq 0$, then using (6), (7) and (8) , we obtain

$$(c-b)(AM+AC+MC) = 3b+3c-2bc-4+b(c-b)+(1-b)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMC is 2.

If $3b+3c-2bc-4 < 0$, then similarly,

$$(c-b)(AM+AB+MB) = -3b-3c+2bc+4+c(c-b)+(c-1)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMB is 2. The result follows.

OC108. Determine all functions $f:\mathbb{R}\mapsto\mathbb{R}$ such that

$$2f(x) = f(x+y)+f(x+2y)\;\forall x\in\mathbb{R}, y\in[0,\infty)\,.$$

Originally question 1 from Romania Team Selection Test 2011, Day 1.

Solved by M. Bataille; D. Văcaru; and T. Zvonaru and N. Stanciu. We give the common solution of Titu Zvonaru and Neculai Stanciu.

If we replace y by $2y$ in the given relation we get

$$2f(x) = f(x+2y)+f(x+4y)\;\forall x\in\mathbb{R}, y\in[0,\infty)\,.$$

Therefore, for all $x \in \mathbb{R}, y \in [0, \infty)$ we have

$$f(x+y) = 2f(x) - f(x+2y) = f(x+4y)\,.$$

Replacing x by $x - y$ we get

$$f(x) = f(x+3y)\,\forall x \in \mathbb{R}, y \in [0, \infty)\,.$$

Then, if $a < b$ are any two real numbers, by setting $x = a$ and $y = \frac{b-1}{3}$ we get

$$f(a) = f(b)\,.$$

This proves that f is a constant function. Conversely, it is easy to check that all constant functions satisfy the given condition.

OC109. Let $a_1, a_2, \ldots, a_n, \ldots$ be a permutation of the set of positive integers. Prove that there exist infinitely many positive integers i so that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

Originally question 2 from China Team Selection 2011, test 3 Day 2.

No solution was received to this problem.

OC110. Let G be a graph, not containing K_4 as a subgraph. If the number of vertices is $3k$, with k integer, what is the maximum number of triangles in G?

Originally question 3 from Mongolia National Olympiad 2011, Team Selection Test Day 2.

Solved by Oliver Geupel. We present his solution below.

We prove that the maximum number of triangles is k^3.

First, we give an example of a graph G with k^3 triangles. The set of $3k$ vertices of G is split into three subsets of cardinality k each. Every vertex of G has an edge to every vertex in the two other subsets, but has no edge to any vertex in its own subset. [*Editor's Comment: G is called the complete tri-partite graph and is usually denoted by $K_{k,k,k}$.*] Clearly, G contains k^3 triangles but does not contain K_4 as a subgraph.

It remains to show that k^3 is also an upper bound.

Let G be a graph with $3k$ vertices, not containing K_4 as a subgraph. We claim that the number of triangles in G does not exceed k^3. Our proof is by induction on the number k.

For $k = 0$ the claim is obviously true.

Now let us assume that $k \geq 1$ and that the claim holds true for the number $k - 1$.

If G does not contain a triangle then there is nothing to prove. Otherwise consider any triangle in G. The set of $3k$ vertices of G is split into the subset V containing the three vertices of the triangle and the subset W of the $3(k-1)$ remaining vertices. The set T of triangles in G is split into four subsets

- the set T_1 of triangles with three vertices in V,
- the set T_2 of triangles with three vertices in W,
- the set T_3 of triangles with one vertex in V and two vertices in W and
- the set T_4 of triangles with two vertices in V and one vertex in W.

Clearly, $|T_1| = 1$.

The node set W induces a subgraph H of G which satisfies the induction hypothesis. Therefore, H contains at most $(k-1)^3$ triangles by induction, which implies $|T_2| \leq (k-1)^3$.

Consider a triangle in T_3. If an edge of H would be combined with two distinct vertices in V to form two triangles, then these four vertices would constitute a 4-clique, which is impossible by hypothesis. Hence, every edge of H can occur in at most one triangle in T_3. Turan's theorem states that a K_{r+1}-free graph with n vertices has at most $\dfrac{(r-1)n^2}{2r}$ edges. Thus, H has not more than $3(k-1)^2$ edges, so that $|T_3| \leq 3(k-1)^2$.

Consider a triangle in T_4. If a vertex in W would occur in two distinct triangles in T_4 then this vertex, being combined with the three vertices in V, would constitute a 4-clique, which is impossible by hypothesis. Hence, every vertex in W can occur in at most one triangle in T_4, so that $|T_4| \leq 3(k-1)$.

Putting everything together, we obtain

$$|T| = |T_1| + |T_2| + |T_3| + |T_4| \leq 1 + (k-1)^3 + 3(k-1)^2 + 3(k-1) = k^3,$$

which completes the induction.