OLYMPIAD SOLUTIONS

OC106. Find all the positive integers n for which all the n digit integers containing $n-1$ ones and 1 seven are prime.

Originally question 3 from Macedonia National Olympiad 2011.

Solved by R. Hess; D. E. Manes; and D. Văcaru. We give two solutions.

Solution 1, by David E. Manes.

If $n = 1$ then the number is 7 which is prime.

If $n = 2$ then the only two possibilities are 71 or 17, which are both primes.

We claim that there is no $n \geq 3$ which works. To see this, we look at the remainder of n when divided by 6.

If $n = 6k$ or $n = 6k+3$, the sum of the digits of all these numbers is $n+6$ which is divisible by 3. Therefore, none of the numbers is prime.

If $n = 6k+1$ with $k \geq 1$, then since $111111 = 7*11*13$, it follows that the number

$$7\underbrace{111\ldots1}_{6k}$$

is divisible by 7, and strictly larger than 7. Therefore, it is not prime.

If $n = 6k+2$, then, as $n > 2$ it follows that $k \geq 1$. Then $n = 6k'+8$ where $k' \geq 0$. Then, as 11171111 is a multiple of 7, it follows that

$$11171111\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

If $n = 6k+4$, then since $7111 = 13 * 547$, it follows that

$$7111\underbrace{111\ldots1}_{6k}$$

is divisible by 13, hence not prime.

Finally, if $n = 6k+5$, then since 11711 is a multiple of 7, it follows that

$$11711\underbrace{111\ldots1}_{6k'}$$

is divisible by 7, and therefore is not prime.

Solution 2, by Richard I. Hess.

It is easy to see that $n = 1, n = 2$ work.

Case $n = 3$ doesn't work because all numbers are divisible by 9.

Case $n = 4$ doesn't work because $1711 = 29 * 59$.

Case $n = 5$ doesn't work because $11711 = 7 * 1763$.

We now prove that no $n \geq 6$ works. First, let us observe that for each n every number has the form

$$\underbrace{111\ldots1}_{n} + 6 * 10^k \,,$$

for some $1 \leq k \leq n$. Next, note that

$$7|\underbrace{111\ldots1}_{n} \quad \text{if and only if} \quad 7|\underbrace{999\ldots999}_{n} \quad \text{if and only if} \quad 7|10^n - 1.$$

This is equivalent to $10^n \equiv 1 \pmod 7 \Leftrightarrow 6 = \text{ord}_7(10)|n$. We split now the problem in two cases.

Case 1: $6|n$. Here all the numbers are divisible by 3, and hence none is prime.

Case 2: $6 \nmid n$. In this case, we saw above that $7 \nmid \underbrace{111\ldots1}_{n}$. Therefore, $\underbrace{111\ldots1}_{n}$ is invertible modulo 7. As 10 is a primitive root modulo 7, there exists a $1 \leq k \leq 6 \leq n$ such that

$$\underbrace{111\ldots1}_{n} \equiv 10^k \pmod 7 \,,$$

Therefore, for this k, 7 divides $\underbrace{111\ldots1}_{n} + 6 * 10^k$ and hence this number is not prime.

It follows that $n = 1$ or $n = 2$.

OC107.

ABC is a triangle of perimeter 4. Point X is marked on the ray AB and point Y is marked on the ray AC such that $AX = AY = 1$. BC intersects XY at M. Prove that one of the triangles ABM or ACM has perimeter 2.

Originally question 4 from Russia National Olympiad 2012, Grade 10 Day 1.

Solved by Michel Bataille whose solution we present below.

Let $BC = a, CA = b, AB = c$. From the hypotheses, $a = 4 - b - c$ and

$$\overrightarrow{AX} = \frac{1}{c}\overrightarrow{AB}, \quad \overrightarrow{AY} = \frac{1}{b}\overrightarrow{AC}\,.$$

In real coordinates relative to (A, B, C), we have

$$X = (c-1:1:0), \quad Y = (b-1:0:1), \quad XY : x + y(1-c) + z(1-b) = 0$$

so that $M = (0 : 1-b : c-1)$. Since M is interior to segment BC, $1-b$ and $c-1$ must have the same sign.

Without loss of generality, we suppose that $c > 1$ and $b < 1$ in what follows. Then

$$(c-b)\overrightarrow{BM} = (c-1)\overrightarrow{BC}$$

and

$$(c-b)\overrightarrow{CM} = (1-b)\overrightarrow{CB},$$

so that

$$(c-b)BM = (c-1)a = (c-1)(4-b-c), \tag{6}$$
$$(c-b)CM = (1-b)a = (1-b)(4-b-c)\,. \tag{7}$$

Also,

$$(c-b)\overrightarrow{AM} = (1-b)\overrightarrow{AB} + (c-1)\overrightarrow{AC}$$

so that

$$(c-b)^2AM^2 = (1-b)^2c^2 + (c-1)^2b^2 + (1-b)(c-1)(2\overrightarrow{AB}\cdot\overrightarrow{AC})$$

with

$$2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2 + c^2 - a^2 = b^2 + c^2 - (4-b-c)^2 = 8b + 8c - 2bc - 16\,.$$

A short calculation gives $(c-b)^2AM^2 = (3b+3c-2bc-4)^2$, hence

$$(c-b)AM = |3b+3c-2bc-4|\,. \tag{8}$$

Now, if $3b+3c-2bc-4 \geq 0$, then using (6), (7) and (8) , we obtain

$$(c-b)(AM+AC+MC) = 3b+3c-2bc-4+b(c-b)+(1-b)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMC is 2.

If $3b+3c-2bc-4 < 0$, then similarly,

$$(c-b)(AM+AB+MB) = -3b-3c+2bc+4+c(c-b)+(c-1)(4-b-c) = 2(c-b)\,,$$

and the perimeter of AMB is 2. The result follows.

OC108. Determine all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$2f(x) = f(x+y) + f(x+2y)\; \forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Originally question 1 from Romania Team Selection Test 2011, Day 1.

Solved by M. Bataille; D. Văcaru; and T. Zvonaru and N. Stanciu. We give the common solution of Titu Zvonaru and Neculai Stanciu.

If we replace y by $2y$ in the given relation we get

$$2f(x) = f(x+2y) + f(x+4y)\; \forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Therefore, for all $x \in \mathbb{R}, y \in [0, \infty)$ we have

$$f(x+y) = 2f(x) - f(x+2y) = f(x+4y)\,.$$

Replacing x by $x - y$ we get

$$f(x) = f(x+3y)\,\forall x \in \mathbb{R}, y \in [0,\infty)\,.$$

Then, if $a < b$ are any two real numbers, by setting $x = a$ and $y = \frac{b-1}{3}$ we get

$$f(a) = f(b)\,.$$

This proves that f is a constant function. Conversely, it is easy to check that all constant functions satisfy the given condition.

OC109. Let $a_1, a_2, \ldots, a_n, \ldots$ be a permutation of the set of positive integers. Prove that there exist infinitely many positive integers i so that $\gcd(a_i, a_{i+1}) \leq \frac{3}{4}i$.

Originally question 2 from China Team Selection 2011, test 3 Day 2.

No solution was received to this problem.

OC110. Let G be a graph, not containing K_4 as a subgraph. If the number of vertices is $3k$, with k integer, what is the maximum number of triangles in G?

Originally question 3 from Mongolia National Olympiad 2011, Team Selection Test Day 2.

Solved by Oliver Geupel. We present his solution below.

We prove that the maximum number of triangles is k^3.

First, we give an example of a graph G with k^3 triangles. The set of $3k$ vertices of G is split into three subsets of cardinality k each. Every vertex of G has an edge to every vertex in the two other subsets, but has no edge to any vertex in its own subset. [*Editor's Comment: G is called the complete tri-partite graph and is usually denoted by $K_{k,k,k}$.*] Clearly, G contains k^3 triangles but does not contain K_4 as a subgraph.

It remains to show that k^3 is also an upper bound.

Let G be a graph with $3k$ vertices, not containing K_4 as a subgraph. We claim that the number of triangles in G does not exceed k^3. Our proof is by induction on the number k.

For $k = 0$ the claim is obviously true.

Now let us assume that $k \geq 1$ and that the claim holds true for the number $k - 1$.

If G does not contain a triangle then there is nothing to prove. Otherwise consider any triangle in G. The set of $3k$ vertices of G is split into the subset V containing the three vertices of the triangle and the subset W of the $3(k-1)$ remaining vertices. The set T of triangles in G is split into four subsets

- the set T_1 of triangles with three vertices in V,
- the set T_2 of triangles with three vertices in W,
- the set T_3 of triangles with one vertex in V and two vertices in W and
- the set T_4 of triangles with two vertices in V and one vertex in W.

Clearly, $|T_1| = 1$.

The node set W induces a subgraph H of G which satisfies the induction hypothesis. Therefore, H contains at most $(k-1)^3$ triangles by induction, which implies $|T_2| \leq (k-1)^3$.

Consider a triangle in T_3. If an edge of H would be combined with two distinct vertices in V to form two triangles, then these four vertices would constitute a 4-clique, which is impossible by hypothesis. Hence, every edge of H can occur in at most one triangle in T_3. Turan's theorem states that a K_{r+1}-free graph with n vertices has at most $\dfrac{(r-1)n^2}{2r}$ edges. Thus, H has not more than $3(k-1)^2$ edges, so that $|T_3| \leq 3(k-1)^2$.

Consider a triangle in T_4. If a vertex in W would occur in two distinct triangles in T_4 then this vertex, being combined with the three vertices in V, would constitute a 4-clique, which is impossible by hypothesis. Hence, every vertex in W can occur in at most one triangle in T_4, so that $|T_4| \leq 3(k-1)$.

Putting everything together, we obtain

$$|T| = |T_1| + |T_2| + |T_3| + |T_4| \leq 1 + (k-1)^3 + 3(k-1)^2 + 3(k-1) = k^3,$$

which completes the induction.