Ramsey's Theorem Through Examples, Exercises, and Problems: Part I

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1 Introduction

Ramsey theory is a contemporary mathematical field that is part of combinatorics. There are applications of Ramsey theory in number theory, geometry, topology, set theory, logic, ergodic theory, information theory, and theoretical computer science. In the words of Imre Leader [3],

> The fundamental kind of question Ramsey theory asks is: can one always find order in chaos? If so, how much? Just how large a slice of chaos do we need to be sure to find a particular amount of order in it?

The starting point in studying Ramsey theory is the 'pigeonhole principle':

Theorem 1 *Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$, then at least one pigeonhole contains at least two pigeons. More generally, there is at least one pigeonhole containing at least $\lceil n/k \rceil$ pigeons.*

Exercise 1 *Use the pigeonhole principle to prove that, for any natural number n, if $a_1, a_2, \ldots, a_{n+1}$ are distinct natural numbers between 1 and $2n$, then there exist i, j, $i \neq j$, such that a_i divides a_j.*

Exercise 1 is known as one of Erdős's favourite questions to ask of an ε. [1]

In Ramsey theory, it is often possible to state difficult problems in a way that any numerically literate person can understand them. As an example, here is a long standing open problem [2]:

Problem 1 *If the set of natural numbers is partitioned in a finite number of cells, must there exist x, y (with x and y not both equal to 2) such that $x + y$ and xy belong to the same cell?*

Ramsey theory is named after British mathematician, economist, and philosopher Frank Ramsey. He was born in 1903 in Cambridge, England, into a family of a Cambridge mathematics professor. The oldest of four siblings, Ramsey married when he was 22 years old and had two daughters. Ramsey died in 1930 at the age of 27. His youngest sister, Margaret Paul [4], suggested that the probable cause of his death was a liver illness brought on by the Hepatitis B virus that Ramsey contracted while swimming in the River Cam. Ramsey was a lifelong literature and music enthusiast and he enjoyed hiking during his vacations. Ramsey significantly

[1] In mathematics, the Greek letter ε is often used to denote a small positive real number. Paul Erdős, a famous Hungarian mathematician and the father of Ramsey theory, used to call young people "epsilons".

contributed to the fields of mathematics, economics, and philosophy while only in his twenties.

2 Ramsey's Theorem

As an introduction to Ramsey's theorem, we look at the following exercises. Consider the global population at the present and imagine that you can form all possible groups of, for example, 10 people. Next, partition this newly formed set of groups of 10 people in, for example, 100 mutually disjoint cells following any criterion you prefer.

Exercise 2 *Use the website called Worldometers* [2] *to find the estimate of the size of the current global population. Call this estimate* m.

Exercise 3 *Use your estimate* m *from Exercise 2 to find the number of different groups of 10 people. Call this number* g_{10} *and write it in scientific notation.*

Exercise 4 *Use your estimate* m *from Exercise 2 to find the number of different groups of 10 people that any given person would belong to. Write your answer in scientific notation.*

Exercise 5 *Use the number* g_{10} *from Exercise 3 to find the number of ways in which you can partition the set of groups of 10 people in 100 mutually disjoint cells. Approximate your answer with a power of* 10.

The enormous size of the number obtained in Exercise 5 illustrates what Leader meant when he asked "can one always find order in chaos?" Is there a pattern that is unavoidable regardless in which way we partition the set of groups of 10 people in 100 mutually disjoint cells? For example,

Question 1 *Can you be sure that for any fixed partition there would be 1000 people so that all groups of 10 that contain only individuals from those chosen 1000 people belong to the same partition cell?*

The answer to Question 1 is, "Yes, if there were enough people on Earth," since according to *Ramsey's Theorem*:

Theorem 2 *Given any* r, n, *and* μ *we can find an* m_0 *such that, if* $m \geq m_0$ *and the* r*-combinations of any* Γ_m *are divided in any manner into* μ *mutually exclusive classes* C_i $(i = 1, 2, \ldots, \mu)$, *then* Γ_m *must contain a sub-class* Δ_n *such that all the* r*-combinations of members of* Δ_n *belong to the same* C_i.

In our example $r = 10$, $n = 1000$, $\mu = 100$, and m represents the size of the global population Γ_m at a certain moment in time. The phrase "r-combinations" in the theorem matches our phrase "the set of groups of 10 people". Moreover, the phrase "a sub-class Δ_n such that all the r-combinations of members of Δ_n belong to the same C_i" refers to our "1000 people so that all groups of 10 that contain

[2] `http://www.worldometers.info/world-population/`

only individuals from those chosen 1000 people belong to the same partition cell". For these values of r, n, and μ, the value of the number m_0 is unknown, but almost certainly the world population will never reach the required m_0.

The above theorem by Frank Ramsey appears in *On a Problem of Formal Logic* in the Proceedings of the London Mathematical Society in 1930 [5]. Ron Graham and Bruce Rothschild, pioneers in Ramsey theory, described Ramsey's theorem in the following way [1]:

> The theorem is a profound generalization of the 'pigeonhole principle' or 'Dirichlet box principle'. As is the case with many beautiful ideas in mathematics, Ramsey's theorem extends just the right aspect of an elementary observation and derives consequences which are extremely natural although far from obvious.

Exercise 6 *For which values of r and n does Ramsey's theorem become the 'pigeonhole principle'?*

3 Ramsey's Theorem: Friends and Strangers

Consider the following so-called 'dinner party problem':

Problem 2 *How many people must be at dinner to ensure that there are either three mutual acquaintances or there are three mutual strangers?*

Exercise 7 *For which values of r, n, and μ does Ramsey's theorem become the 'party problem'?*

To solve Problem 2, we use so-called *edge 2-colourings of complete graphs* on 5 and 6 vertices, K_5 and K_6. The complete graph K_m on m vertices is represented by a drawing in which we first draw m points in the plane so that no three points are collinear and then we draw a line segment between each pair of those m points. The points are called *vertices* and the line segments are called *edges* (Figure 1).

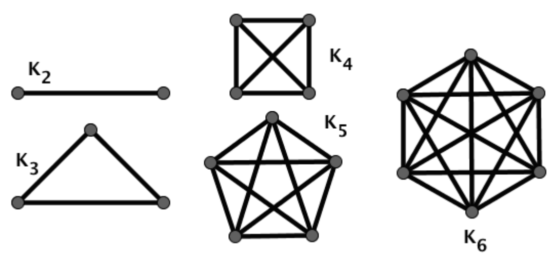

Figure 1: Complete graphs K_2, K_3, K_4, K_5 and K_6

Exercise 8 *Use two colours, say red and blue, to colour the edges of K_6. Each edge can be coloured by only one colour. (In more mathematical terms, you are asked to perform an edge 2-colouring of K_6.)*

Note that in Exercise 8 you have two colours available, but you may wish to use only one colour.

Question 2 *How many different edge 2-colourings of* K_6 *are there?*

Question 3 *Can you find a monochromatic triangle in your colouring; i.e., can you find three edges coloured by the same colour that form a triangle? (Note that any triangle represents* K_3*.)*

Problem 3 *Does any edge 2-colouring of* K_6 *yield a monochromatic* K_3*?*

Exercise 9 *Explain in a short paragraph why Problem 2 and Problem 3 are equivalent.*

The answer to Problem 3 is, "Yes, it does." To see this, follow these two steps.

> **Step 1:** Suppose that you are given an edge 2-colouring of K_6. Fix one vertex. How many edges are coming out of this vertex? Based on which theorem can you conclude that at least three of those edges will be of the same colour?
>
> **Step 2:** Consider all possible 2-colourings of the triangle determined by the vertices adjacent by the three edges of the same colour to the originally fixed vertex.

Exercise 10 *Find an edge 2-colouring of* K_5 *with no monochromatic triangles.*

Exercise 11 *Based on Exercises 7, 9, and 10 and the answer to Problem 3 conclude that, in the notation of Ramsey's theorem, if* $r = \mu = 2$ *and* $n = 3$ *then* $m_0 = 6$*. This fact is usually written* $R(3,3) = 6$ *with the meaning that any 2-colouring of* K_6 *yields a monochromatic* K_3 *in the 'first' colour or a monochromatic* K_3 *in the 'second' colour and that there is an edge 2-colouring of* K_5 *with no monochromatic triangles.*

In general, for natural numbers s and t, $s, t \geq 2$, we define *the Ramsey number* $R(s,t)$ as the minimum number n for which any edge 2-colouring of K_n in red and blue contains a red K_s or a blue K_t.

Exercise 12 *Find* $R(2,t)$ *for any natural number* $t \geq 2$*. What can you tell about* $R(t,2)$*?*

Problem 4 *Any graph with at least 6 vertices contains a complete subgraph on 3 vertices or an independent subgraph of 3 vertices (An independent subgraph of a given graph consist of vertices of which no pair is adjacent).*

Problem 5 *True or False: Each 2-colouring of* K_6 *yields at least two monochromatic triangles?*

Hints and comments

Exercise 1 For each i, write a_i as $a_i = 2^{b_i}q_i$, where q_i is an odd number, and consider the sequence of odd numbers $\{q_1, \ldots, q_{n+1}\}$ in $[1, 2n]$.

Exercise 3 Note that $g_{10} = \binom{m}{10}$. Write this number in scientific notation.

Exercise 4 Note that this number is given by $\binom{m-1}{9}$.

Exercise 5 There are 100 cells and for each cell there are $G = \binom{g_{10}}{10}$ choices. Thus, the number of ways is $100^G = 10^{2G}$. Use Exercise 4 to complete this exercise.

Problem 4 Suppose that a graph G with more than 6 vertices is given. Colour all edges of the graph G red. Next, draw all missing edges and colour them blue.

Problem 5 There are 20 different triangles in K_6. Colour the edges of K_6 with red and blue and call a triangle in K_6 *2-coloured* if it is not monochromatic. Each 2-coloured triangle contains two *2-coloured* angles, i.e., two angles with sides of different colours. Conclude that the number of 2-coloured triangles is equal to one half of the number of 2-coloured angles. Count the number of the 2-coloured angles to see that the number of 2-coloured triangles is at most 18.

References

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