

Excerpt from The Math Olympian

Richard Hoshino

Editor's Prologue. *The Math Olympian* is a novel by Richard Hoshino, himself a former Olympian who today teaches mathematics at Quest University Canada. The story traces the growth of Bethany MacDonald from an insecure and bullied grade 5 student, who is unhappy in school, to a confident high schooler who is about to write the Canadian Mathematical Olympiad and maybe realize her dream of qualifying for the Canadian IMO team. The story begins as Bethany looks at the first problem shortly after the contest starts, at 9:00 a.m.

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The Canadian Mathematical Olympiad, Problem #1:

Determine the value of:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}} + 3} + \frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}} + 3} + \frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}} + 3} + \cdots + \frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}} + 3} + \frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}} + 3}.$$

I stare at the first problem, not sure where to start.

I circle the first term in the expression of Problem #1, the one with the ugly exponent $9^{\frac{1}{1000}}$. Am I actually supposed to calculate the 1000th root of 9? Without a calculator, I know that's not possible.

There has to be an insight somewhere. This is an Olympiad problem, and all Olympiad problems have nice solutions that require imagination and creativity rather than a calculator.

I re-read the question yet again, and confirm that I have to determine the following sum:

$$\frac{9^{\frac{1}{1000}}}{9^{\frac{1}{1000}} + 3} + \frac{9^{\frac{2}{1000}}}{9^{\frac{2}{1000}} + 3} + \frac{9^{\frac{3}{1000}}}{9^{\frac{3}{1000}} + 3} + \cdots + \frac{9^{\frac{998}{1000}}}{9^{\frac{998}{1000}} + 3} + \frac{9^{\frac{999}{1000}}}{9^{\frac{999}{1000}} + 3}.$$

There are 999 terms in the sum, and each term is of the form $\frac{9^x}{9^x + 3}$. In the first term, x equals $\frac{1}{1000}$; in the second term, x equals $\frac{2}{1000}$; in the third term, x equals $\frac{3}{1000}$; and so on, all the way up to the last term, where x equals $\frac{999}{1000}$.

In the entire expression, there's only one doable calculation, the term right in the middle. I know I can calculate $\frac{9^{\frac{500}{1000}}}{9^{\frac{500}{1000}} + 3}$, using the fact that $\frac{500}{1000} = \frac{1}{2}$.

Since raising a quantity to the exponent $\frac{1}{2}$ is the same as taking its square root, I see that:

$$\frac{9^{\frac{500}{1000}}}{9^{\frac{500}{1000}} + 3} = \frac{9^{\frac{1}{2}}}{9^{\frac{1}{2}} + 3} = \frac{\sqrt{9}}{\sqrt{9} + 3} = \frac{3}{3 + 3} = \frac{3}{6} = \frac{1}{2}.$$

But other than this, I'm not sure what to do. Twirling my pen and closing my eyes, I concentrate, hoping for a spark.

One idea comes to mind: setting up a “telescoping series”. My mentor, Mr. Collins, introduced me to this beautiful technique years ago at one of our Saturday afternoon sessions at Le Bistro Café. Before explaining the concept to me, Mr. Collins first gave me a simple question of adding five fractions:

$$\text{Without using a calculator, determine } \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30}.$$

I solved Mr. Collins problem by finding the common denominator. In this case, the common denominator is 60, the smallest number that evenly divides into each of 2, 6, 12, 20, and 30. So the answer is:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{30 + 10 + 5 + 3 + 2}{60} = \frac{50}{60} = \frac{5}{6}.$$

And then I remembered Mr. Collins' smile as he gave me another addition problem:

$$\text{Determine } \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90}$$

This time, it took me almost fifteen minutes to get the answer. Most of the time was spent trying to figure out the common denominator, which I eventually determined to be 2520. But it was a tedious process of checking and re-checking all of my calculations.

After Mr. Collins congratulated me on getting the right answer, he pointed to the nine fractions on my sheet of paper and asked if there was a pattern. After staring at the numbers for a while, I saw it:

$$\begin{array}{lll} \mathbf{2} & = & 1 \times 2 & \mathbf{6} & = & 2 \times 3 & \mathbf{12} & = & 3 \times 4 \\ \mathbf{20} & = & 4 \times 5 & \mathbf{30} & = & 5 \times 6 & \mathbf{42} & = & 6 \times 7 \\ \mathbf{56} & = & 7 \times 8 & \mathbf{72} & = & 8 \times 9 & \mathbf{90} & = & 9 \times 10 \end{array}$$

Mr. Collins suggested I write $\frac{1}{90}$ as the difference of two fractions: $\frac{1}{90} = \frac{1}{9} - \frac{1}{10}$. He then asked whether there were any other terms in this expression that could also be written as the difference of two fractions. I eventually saw that $\frac{1}{2} = \frac{1}{1} - \frac{1}{2}$ and $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$.

Once I saw the pattern, I discovered this amazing solution, called a “telescoping series”: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90}$ can be re-written as:

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right).$$

This is just $\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{8} + \frac{1}{8} - \frac{1}{9} + \frac{1}{9} - \frac{1}{10}$.

Since one negative fraction cancels a positive fraction with the same value, all the terms in the middle get eliminated:

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{8} + \frac{1}{8} - \frac{1}{9} + \frac{1}{9} - \frac{1}{10}.$$

Like a giant telescope that collapses down to a small part at the top and a small part at the bottom, this series collapses to the difference $\frac{1}{1} - \frac{1}{10}$, which equals $\frac{9}{10}$. So the answer is $\frac{9}{10}$.

That day, Mr. Collins showed me several problems where the answer can be found using a telescoping series, where a seemingly-tedious calculation can be solved with elegance and beauty.

The key is to represent each term as a difference of the form $x - y$, where y is called the “subtrahend” and x is called the “minuend”. From Mr. Collins’ examples, I learned that the series telescopes every time the subtrahend of one term equals the minuend of the following term.

As I recall that lesson with Mr. Collins many years ago, I’m hopeful that I can use this technique to solve the first problem of the Canadian Math Olympiad. I look at Problem #1 again, reminding myself of what I need to determine.

I start with the general expression $\frac{9^x}{9^x+3}$ and try to write it down as the difference of two functions, so that the subtrahend of each term equals the minuend of the following term.

I try a bunch of different combinations to the difference to work out to $\frac{9^x}{9^x+3}$ such as the expression $\frac{1}{3^x} - \frac{1}{3^{x+1}}$ which almost works but not quite. I attempt other combinations using every algebraic method I know. All of a sudden, I realize the futility of my approach.

The denominator doesn’t factor nicely, so this approach cannot work. Oh no.

9:19 a.m.

I feel the first bead of sweat on my forehead, and wonder if I’m going to get another “math contest anxiety attack”. I close my eyes and take a deep breath, knowing that if I start to panic and lose focus, my chances of becoming a Math Olympian are over.

Calm down, Bethany, calm down. There’s lots of time left. You can do this.

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To be continued in issue 4.

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