CONTEST CORNER SOLUTIONS

CC56. From the set of consecutive integers $\{1, 2, 3, \ldots, n\}$, three integers that form a geometric sequence are deleted. The sum of the integers remaining is 6125. Determine the smallest value of n and all three-term geometric sequences that make this possible.

Originally 1996 Invitational Mathematics Challenge, Grade 11, problem 5.

We present the solution by Konstantine Zelator.

We know that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ and that a, ar, ar^2 are the terms of a geometric sequence. Thus $a + ar + ar^2 + 6125 = \frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2} > 6125$. The smallest value of n which works is 111. This means that $a(1+r+r^2) = 91$. We have two cases.

Case 1 : r is a positive integer.

Then, both a and r^2+r+1 are natural numbers. Since 91=7×13, we get the following : $a = 1, r = 9$, or $a = 7, r = 3$, or $a = 13, r = 2$.

Case 2 : $r > 1$ and r is a fraction.

This means that $r = \frac{d}{c}$, where c and d are relatively prime positive integers with $c \geq 2$ and $d \geq 3$. Then $a(1 + \frac{d}{c} + \frac{d^2}{c^2}) = 91$ or $a(d^2 + cd + c^2) = 91c^2$.

Since ar^2 is an integer, we know $\frac{ad^2}{c^2}$ is an integer and because c does not divide d, c^2 divides a. Let $a = c^2k$ for some positive integer k. Then $k(d^2+cd+c^2) = 91$ and since we know $c \geq 2$ and $d \geq 3$ giving us $d^2+cd+c^2 \geq 19$. Thus $k = 1$ and $d^2 + cd + c^2 = 91$. The only solution is $d = 6$ and $c = 5$ and $a = 25$.

Therefore there are four 3-term sequences that satisfy the conditions :

$$1, 9, 81 \quad 7, 21, 63 \quad 13, 26, 52 \quad 25, 30, 36.$$

CC57. Triangle DEF is acute. Circle C_1 is drawn with DF as its diameter and circle C_2 is drawn with DE as its diameter. Points Y and Z are on DF and DE respectively so that EY and FZ are altitudes of ΔDEF. EY intersects C_1 at P, and FZ intersects C_2 at Q. EY extended intersects C_1 at R, and FZ extended intersects C_2 at S. Prove that P, Q, R, and S are concyclic points.

Originally 2002 Canadian Open Mathematics Challenge, problem B4.

Solved by S. Muralidharan ; and Z. Burnett. We present the solution by S. Muralidharan.

We will show that the points P, Q, R and S lie on a circle with centre D.

Let $\angle EDF$ be denoted by D, length $DF = y$ and length $DE = z$. Since DF is the diameter of the circle C_1 and EY is perpendicular to DF, it follows that $DP = DR$. Now, $DY = z\cos D$ and $O_1P = O_1D = \frac{y}{2}$. From the right-angled triangle PYO_1, we get :

$$PY^2 = O_1P^2 - O_1Y^2 = \frac{y^2}{4} - \left(\frac{y}{2} - z\cos D\right)^2 = yz\cos D - z^2\cos^2 D.$$

From right-angled triangle DPY, we have :

$$DP^2 = PY^2 + DY^2 = yz\cos D - z^2\cos^2 D + z^2\cos^2 D = yz\cos D.$$

Thus, we have $DP = DR = yz\cos D$.

By symmetry, if we use the above argument with the circle C_2, we get

$$DQ = DS = yz\cos D.$$

Thus P, Q, R and S lie on a circle with centre D and radius $yz\cos D$.

CC58. Find all real values of x, y and z such that

$$\begin{aligned} x - \sqrt{yz} &= 42 \\ y - \sqrt{zx} &= 6 \\ z - \sqrt{xy} &= -30\,. \end{aligned}$$

Originally problem B4 of 1997 Canadian Open Mathematics Challenge.

Solved by Š. Arslanagić ; M. Coiculescu ; J. L. Díaz-Barrero ; D. Văcaru ; E. Wang ; K. Zelator ; and T. Zvonaru. We present the solution of Titu Zvonaru.

The first and second equation imply $x, y > 0$, then from $xz > 0$, we conclude $z > 0$. Hence we make a substitution $x = a^2, y = b^2, z = c^2$. Our system becomes

$$a^2 - bc = 42, \quad b^2 - ac = 6, \quad c^2 - ab = -30. \tag{1}$$

Subtracting the second equation from the first equation, and subtracting the third from the second gives us the following two equations :

$$(a-b)(a+b+c) = 36, \tag{2}$$

$$(b-c)(a+b+c) = 36. \tag{3}$$

Hence $a-b, b-c$ and $a+b+c$ are all non-zero and $a-b = b-c$. This implies $a = 2b - c$. Substituting this into (1) yields

$$4b^2 - 5bc + c^2 = 42, \tag{4}$$

$$b^2 - 2bc + c^2 = 6. \tag{5}$$

From (5), $b - c = \pm\sqrt{6}$, and hence from (4), $c = \pm\sqrt{6}$. We then conclude the solutions to the system (1) are

$$(-3\sqrt{6}, -2\sqrt{6}, -\sqrt{6}),\ (3\sqrt{6}, 2\sqrt{6}, \sqrt{6}),$$

which each yield $x = 54, y = 24, z = 6$.

CC59. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place ?

Originally Question 3 of 2013 Stanford Math Tournament, Team test.

One incorrect solution was received.

CC60. How many integer solutions are there to

$$a_0^2 + a_0a_1 + a_1^2 + a_1a_2 + \cdots + a_{2009}a_{2010} + a_{2010}^2 = 1?$$

Originally 2010 APICS Math Competition, Question 5.

Solved by Richard Hess, whose solution we present below.

We consider the more general equation, where 2010 is replaced by an arbitrary n. Then, multiplying the equation by 2, we get

$$(0 + a_0)^2 + (a_0 + a_1)^2 + (a_1 + a_2)^2 + \cdots + (a_{n-1} + a_n)^2 + (a_n + 0)^2 = 2.$$

Define $b_0 = a_0$, $b_1 = a_0 + a_1$, $\cdots$, $b_k = a_{k-1} + a_k$, $\cdots$, $b_{n+1} = a_n$. Since our a_i are integers, so will the b_i. It follows that exactly two of the b_i will be nonzero. There are $\frac{(n+1)(n+2)}{2}$ many ways to choose $k < \ell$ so that $b_k, b_\ell \neq 0$.

Notice that $b_k^2, b_\ell^2 = 1$ so $b_k = \pm 1$. Notice that once we choose a_k then all other a_i are decided : $a_0 = a_1 = \cdots = a_{k-1} = 0$, $a_k = -a_{k+1} = \cdots = (-1)^{\ell-k-1}a_{\ell-1}$, $a_\ell = \cdots = a_n = 0$. It follows then that $a_k = \pm 1$, so there are only two possible choices for a_k. So the total number of solutions is $N = 2\frac{(n+1)(n+2)}{2} = (n+1)(n+2)$.

Since $n = 2010$, we see that $N = 2011 \cdot 2012 = 4046132$.