PROBLEM SOLVER'S TOOLKIT

No. 6

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The Problem Solver's Toolkit is a new feature in ***Crux Mathematicorum****. It will contain short articles on topics of interest to problem solvers at all levels. Occasionally, these pieces will span several issues.*

Harmonic Sets Part 3: The Harmonic Mean File

Murray Klamkin, problem solver extraordinaire, was closely associated with *Crux Mathematicorum* during its first, and his last, 30 years. Well into his 70s he claimed that he could remember every result he ever proved, plus its proof. For those youngsters who are reading this, let me warn you that Klamkin was an exception. I realized the need for a good filing system when still in my 30s. The result that convinced me of my vulnerability was one that at first sight I found hard to believe:

> *If AA' and BB' are two line segments that are on the same side of the line AB and perpendicular to it, then the distance d to the line from the point where AB' intersects $A'B$ is independent of the distance AB.*

I easily found a quick argument to show that not only is the result "obvious", but $d = \dfrac{AA' \cdot BB'}{AA' + BB'}$. Unfortunately, a few years later I could not remember my neat proof and had to resort to a proof by algebra. Afterwards, after much effort, I finally recovered my original argument and filed it away for safe keeping. Before turning to that proof, here are a couple other lessons about filing that I learned the hard way: when an item fits in more than one place, put a note in each relevant file indicating the file where that item is located; also, list the contents of each file on its cover. In these final two installments of the four-part series we will look at the items in my first file, the harmonic-mean file.

Now for the proof that d is independent of the distance between A and B. Figure 1, on the next page, almost says it all.

Note that the line A^*B^* parallel to $A'B$ through B' determines a triangle A^*AB^* that is similar to triangle $A'AB$; the dilatation that shrinks the larger to the smaller takes $B'B$ to $D'D$ and A^*A to $A'A$, whence $\dfrac{d}{b} = \dfrac{a}{a+b}$, or

$$d = \frac{ab}{a+b},$$

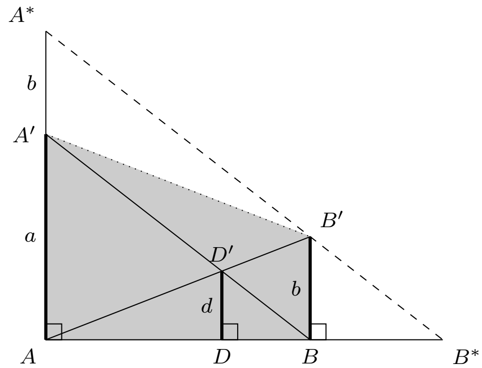

Figure 1: $d = \dfrac{ab}{a+b}$.

which is half the harmonic mean of a and b and is independent of the distance AB, as claimed.

Dictionaries tell us that the *harmonic mean* h of the numbers a and b equals the reciprocal of the arithmetic mean of the reciprocals of a and b. What a mouthful! It is perhaps less formidable in symbols:

$$\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \qquad \text{or} \qquad h = \frac{2ab}{a+b}.$$

The harmonic mean arises naturally when finding average rates. For example, if one pedals a bicycle up the hill at 8 km/h and back down at 24 km/h, then the average speed for the return trip is 12 km/h, the harmonic mean of 8 and 24. Almost the same problem, except here we want the combined rate rather than the average: if it takes 8 minutes for person A to peel the potatoes, and 24 for person B, then how long would it take them if they worked together? No, not 32 minutes, nor 16, but 6 minutes, which is *half* the harmonic mean. As we go through my harmonic-mean file, we shall see that in geometry the harmonic mean pops up all over the place.

Looking carefully at the argument based on Figure 1, we see that we never used perpendicularity — we require only that AA', BB', and DD' be parallel. Moreover, if you complete the trapezoid $A'ABB'$ of Figure 1 you get the theorem that

> *the line that is parallel to the bases of a trapezoid and that passes through the intersection of its diagonals is intercepted by the nonparallel sides in a segment whose length is the harmonic mean of the bases.*

The independence of d (in the result discussed at the start) can also be easily seen dynamically — without computing its value — with the help of a strain whose axis is BB' and whose centre is the point at infinity of the line AB. Recall (from the first installment) that a strain is the perspective collineation that fixes all points of the axis and slides points along lines through the centre. As in Figure 2, the points A, A', D, D' slide along horizontal lines to $\hat{A}, \hat{A}', \hat{D}, \hat{D}'$ while the lengths along the segments parallel to BB' remain constant: BB' is fixed while $AA' = \hat{A}\hat{A}'$ and $DD' = \hat{D}\hat{D}'$.

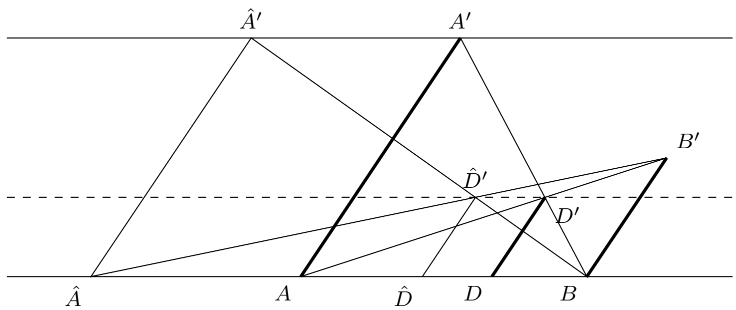

Figure 2: Fix B and B' while letting A and A' move along their horizontal lines to positions $\hat{A}$ and $\hat{A}'$. The distance of the intersection point $\hat{D}'$ from the line AB is independent of the choice of $\hat{A}$.

Next, consider four collinear points A, B, C, and D. We saw in the second installment that B and D are harmonic conjugates with respect to A and C (and the four points form a harmonic set) if and only if the cross ratio $\dfrac{AB \cdot CD}{AD \cdot CB}$ equals -1. A simple calculation shows that for points in the Euclidean plane, the length of AC is the harmonic mean of the segments AB and AD if and only if B and D are harmonic conjugates with respect to A and C:

$$\begin{aligned} AC &= \frac{2AB \cdot AD}{AB + AD} \\ 2AB \cdot AD &= AC \cdot (AB + AD) \\ AB \cdot (AD - AC) &= AD \cdot (AC - AB) \\ AB \cdot CD &= AD \cdot BC = -AD \cdot CB \end{aligned}$$

We conclude this month's installment with a further look at Figure 1. It can be recognized as the initial step of the affine version of Figure 3.5A in [1, p. 32], which indicates how to construct a *harmonic sequence* $A_1, A_2, A_3 \ldots$ from three collinear points O, A_1, and A_2. For $j > 1$ in the sequence, A_j is the harmonic conjugate of O with respect to A_{j-1} and A_{j+1}; this means that

$$OA_{j+1} \cdot A_j A_{j-1} = A_{j+1} A_j \cdot OA_{j-1},$$

or, if you prefer, OA_j is the harmonic mean of OA_{j+1} and OA_{j-1}. Figure 3 shows the start of the infinite harmonic sequence $A_1, A_2, A_3 \ldots$ along the base OA_1 of the parallelogram $OA_1A_1'P$. The harmonic relationship follows directly from the definition of harmonic sets (as given in the previous installment): For all $j > 1$, points O and A_j are diagonal points of the quadrangle whose vertices are P, A'_{j+1}, A'_j, and the common point at infinity of the lines OP and $A_jA'_j$; the remaining diagonals meet OA_j at A_{j-1} and A_{j+1}.

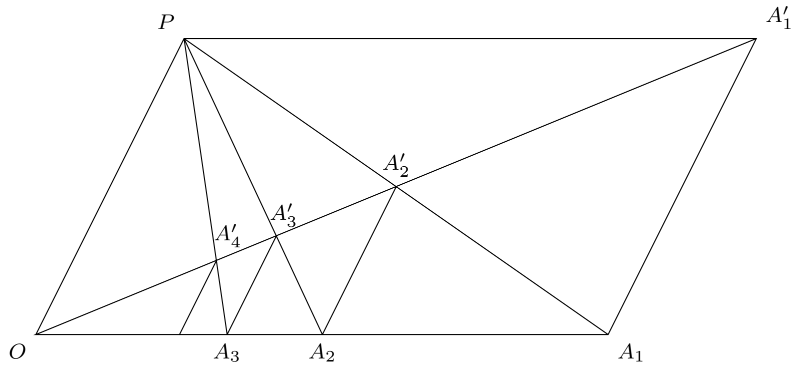

Figure 3: Construction of the harmonic sequence $A_1, A_2, A_3 \ldots$.

Returning to algebra, we see that if segment OA_1 has unit length, then the lengths OA_j form the familiar harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$. Coxeter [1, p. 23] explains the source of the word *harmonic* by labeling $C = O, E = A_5$, and $G = A_3$, and observing that "if the segment CA_1 represents a stretched string, tuned to the note C, the same string stopped at E or G will play the other notes of the major triad." Of course, $\frac{1}{3}$ is the harmonic mean of $\frac{1}{5}$ and 1.

In my file along with this example of harmonic sets is an article [3] by two 14-year-olds who discovered the construction of Figure 3 using *The Geometer's Sketchpad*; the article was brought to my attention by the media frenzy purporting that the students' construction had slipped by the notice of mathematicians for millennia [2]. To the contrary, of course, the construction has been widely known for centuries.

References

[1] H.S.M. Coxeter, *Projective Geometry*, 2nd ed. Springer-Verlag, 1987.

[2] Leslie Chess Feller, The Eternal Challenge of Euclid's Geometry, *The New York Times*, March 7, 1999.

[3] Dan Litchfield, Dave Goldenheim, Euclid, Fibonacci, Sketchpad, *Math. Teacher*, **90**:1 (Jan. 1997) 8-12.