SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3751. [2012 : 241, 243] *Proposed by Richard K. Guy, University of Calgary, Calgary, AB.*

The edge lengths of a quadrilateral are $AB = 5$, $BC = 10$, $CD = 11$, $DA = 14$.

(a) If the quadrilateral is cyclic, what is the diameter of its circumcircle?

(b) If we alter the order of the edges, does it affect the answer to (a)?

I. Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA.

For part (a) note that

$$AB^2 + DA^2 = 5^2 + 14^2 = 221 = 10^2 + 11^2 = BC^2 + CD^2.$$

Hence, if we take $BD = \sqrt{221}$, then ABD and BCD will form two right triangles that share their hypotenuse BD, which implies that the resulting quadrilateral $ABCD$ has a circumcircle whose diameter is $BD = \sqrt{221}$.

For part (b) the answer is *no*, altering the order will generally produce a new quadrilateral, but the circumcircle of the new quadrilateral will have the same diameter. To see this, we denote the centre of the circle of part (a) by O. Altering the order of the edges is the same as interchanging the triangles OAB, OBC, OCD, ODA. No matter how these triangles might be permuted, the four angles at O will still sum to $360°$, and the sides opposite O would form the sides of a new quadrilateral that is still inscribed in the circle whose radius is $OA = OB = OC = OD = \frac{\sqrt{221}}{2}$.

II. Solution by John Hawkins and David R. Stone, Georgia Southern University, Statesboro, GA, USA.

We answer both parts together. In the cyclic quadrilateral $ABCD$, let $a = AB = 5$, $b = BC = 10$, $c = CD = 11$, and $d = DA = 14$, and let $s = \frac{1}{2}(a + b + c + d) = 20$ be the semiperimeter. Brahmagupta's formula gives us the area of the cyclic quadrilateral:

$$A = \sqrt{(s-a)(s-b)(s-c))(s-d)} = \sqrt{(20-5)(20-10)(20-11)(20-14)} = 90.$$

Note that it is a symmetric polynomial in the four variables, so that A is invariant with respect to altering the order of the side lengths. Also the product

$$P = (ac + bd)(ad + bc)(ab + cd) = (55 + 140)(70 + 110)(50 + 154) = 195 \cdot 180 \cdot 204$$

is a symmetric polynomial and therefore invariant under reordering of the side lengths. On the *MathWorld* web page for cyclic quadrilaterals [or any other standard reference] we find the formula $4RA = \sqrt{P}$ involving the circumradius R; consequently, the diameter equals

$$2R = \frac{\sqrt{P}}{2A} = \frac{\sqrt{195 \cdot 180 \cdot 204}}{2 \cdot 90} = \sqrt{13 \cdot 17} = \sqrt{221},$$

and it will not change when the order of the edges is altered.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; MARIAN DINCĂ, Bucharest, Romania; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; TITU ZVONARU, Cománeşti, Romania; and the proposer.

We commonly accept convexity to be part of the definition of a cyclic quadrilateral. Both solutions show further that the area of a cyclic quadrilateral will not change when the order of the edges is altered. More precisely, given the four line segments that form a cyclic quadrilateral, they will, in general, in their six possible orders form three convex quadrilaterals that are not congruent, yet they will have the same circumradius and the same area. Only the proposer addressed the corresponding results for crossed quadrilaterals that are inscribed in a circle. The first solution shows that when the edges are not just rearranged, but are allowed to form a crossed quadrilateral, those quadrilaterals will still have the same circumcircle. (The formula for $2R$ in the second solution requires convexity; it should not be used for crossed quadrilaterals. Indeed, the circumradius of a crossed quadrilateral that is inscribed in a circle is generally different from the common circumradius of its convex mates.)

Most of the submissions were similar to one of the featured solutions, although many provided more background details. Such details were discussed recently in the solution of the related problem 2724, which appeared in the March issue [2013 : 148-149].

3752. [2012 : 241, 243] *Proposed by Péter Ivády, Budapest, Hungary.*

Show that if $n \geq 2$ is a positive integer then

$$\frac{1}{2}\left[1 + \frac{1}{n}\left(1 - \frac{1}{n}\right)\right]^2 < \left(1 - \frac{1}{2^3}\right)\left(1 - \frac{1}{3^3}\right)\cdots\left(1 - \frac{1}{n^3}\right)$$

holds.

Solution by Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

We will prove the claim by induction on n.

First, if $n \geq 2$, then the claim holds since

$$\frac{1}{2}\left[1+\frac{1}{2}\left(1-\frac{1}{2}\right)\right]^2 = \frac{25}{32} < \frac{7}{8} = \left(1-\frac{1}{2^3}\right).$$

Assume the claim is true for $n = k$. So we have

$$\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2 < \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\dots\left(1-\frac{1}{k^3}\right) \qquad (1)$$

and we need to show that the claim is true for $n = k+1$. Multiplying inequality (1) by $\left(1-\frac{1}{(k+1)^3}\right)$, we obtain

$$\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2\left(1-\frac{1}{(k+1)^3}\right) < \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\dots\left(1-\frac{1}{k^3}\right)\left(1-\frac{1}{(k+1)^3}\right) \qquad (2)$$

Now, we notice that

$$\frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2\left(1-\frac{1}{(k+1)^3}\right) - \frac{1}{2}\left[1+\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)\right]^2 = \frac{3-11k^2-8k^3+3k^4+2k^5}{2k^3(1+k)^4} = \frac{(2k^2-8)k^3+(3k^2-11)k^2+3}{2k^3(1+k)^4} > 0,$$

since both $2k^2-8 \geq 0$ and $3k^2-11>0$ as $k \geq 2$. Therefore, we have

$$\frac{1}{2}\left[1+\frac{1}{k+1}\left(1-\frac{1}{k+1}\right)\right]^2 < \frac{1}{2}\left[1+\frac{1}{k}\left(1-\frac{1}{k}\right)\right]^2\left(1-\frac{1}{(k+1)^3}\right). \qquad (3)$$

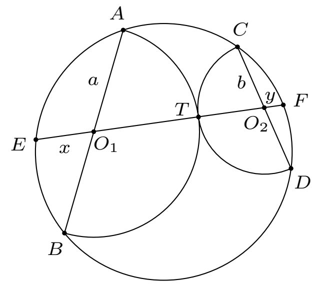
Thus, from (2) and (3) the claim is true for $n = k+1$.

This completes the proof of the original inequality for all $n \geq 2$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ALBERT STADLER, Herrliberg, Switzerland; and the proposer.

3753. [2012 : 241, 243] *Proposed by Abdilkadir Altintaş, mathematics teacher, Emirdağ, Turkey.*

Semi-circles with centres O_1 and O_2 are drawn on chords AB and CD of a circle Γ such that they are tangent at T. The line through O_1 and O_2 intersects Γ at E and F. If $O_1A = a$, $O_2C = b$, $O_1E = x$ and $O_2F = y$, show that $a-b = x-y$.



Solution by several respondents.

By the Intersecting Chords Theorem, we find that $a^2 = x(a + b + y)$ and $b^2 = y(a + b + x)$. Therefore

$$a^2 - b^2 = (x - y)(a + b)$$

from which the result follows.

Solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AN-ANDUUD Problem Solving Group, Ulaanbaatar, Mongolia; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; CAO MINH QUANG, Nguyen Binh Khiem High School, Vinh Long, Vietnam; DAO THANH OAI, Kien Xuong, Thai Binh, Viet Nam; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; IAN JUNE L. GARCES, Ateneo de Manila University, Quezon City, The Philippines; OLIVER GEUPEL, Brühl, NRW, Germany; LEONARD GIUGIUC, Romania; JOHN G. HEUVER, Grande Prairie, AB; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; KEE-WAI LAU, Hong Kong, China; PANAGIOTE LIGOURAS, Leonardo da Vinci High School, Noci, Italy; RICARD PEIRÓ, IES "Abastos", Valencia, Spain; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; JACQUES VERNIN, Marseille, France; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Dao proposed the following generalization for which he enclosed a proof by Lui González: Suppose that the circles with diameters AB and CD do not necessarily intersect and that their radical axis meets EF at T. Then $TO_1 - TO_2 = EO_1 - EO_2$.

3754. [2012 : 242, 243] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that in all scalene triangles ΔABC the inequality

$$576\sqrt{3}r^3 < \frac{w_a^2 - w_b^2}{b - a} + \frac{w_b^2 - w_c^2}{c - b} + \frac{w_c^2 - w_a^2}{a - c} < 72\sqrt{3}R^3$$

holds, where w_a, w_b and w_c are the lengths of the angle bisectors; R is the radius of the circumcircle; and r is the inradius of ΔABC.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The two inequalities do not generally hold. Consider a right triangle with sides $a = 5t$, $b = 4t$, and $c = 3t$ where t is a positive real parameter. Its semiperimeter is $s = 6t$. Straightforward computations yield

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2} = \frac{288}{49}t^2, \qquad w_b^2 = \frac{45}{4}t^2, \qquad w_c^2 = \frac{160}{9}t^2,$$

$$R = \frac{5}{2}t, \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = t.$$

The left inequality thus specializes to $576\sqrt{3}t^3 < \dfrac{2624}{147}t$, which is false when

$$t^2 \geq \frac{2624}{147 \cdot 576\sqrt{3}}.$$

The right inequality rewrites as $\dfrac{2624}{147}t < 1125\sqrt{3}t^3$. But this fails when

$$t^2 \leq \frac{2624}{147 \cdot 1125\sqrt{3}}.$$

Consequently, both inequalities are not generally valid.

One incorrect solution was received.

Upon closer inspection, the proposer lost a factor part way through his solution (as did the person who sent in the incorrect solution). As a result, the original inequality should have read

$$576\sqrt{3}r^3 < \sum_{cyclic} \frac{(w_a^2 - w_b^2)(b+c)^2(a+c)^2}{c(b-a)(a+b+c)} < 72\sqrt{3}R^3$$

which is less appealing to look at than the original. Using Geupel's right triangle example in the inequality above yields

$$576\sqrt{3}t^3 < 1524t^3 < 1125\sqrt{3}t^3$$

which is true.

3755. [2012 : 242, 244] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Find all real numbers $a \leq b \leq c \leq d$ which form an arithmetic progression which satisfy the two equations $a + b + c + d = 1$ and $a^2 + b^2 + c^2 + d^2 = d$.

Solution by Peter Y. Woo, Biola University, La Mirada, CA, USA; and Titu Zvonaru, Comǎneşti, Romania(independently).

Let the four numbers be $a = m - 3h$, $b = m - h$, $c = m + h$, $d = m + 3h$ where $h \geq 0$. The two equations are equivalent to $4m = 1$ and $4m^2 + 20h^2 = m + 3h$. This leads to $20h^2 = 3h$ which implies that $h = 0$ or $h = \frac{3}{20}$. The two solutions are

$$(a, b, c, d) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \left(-\frac{1}{5}, \frac{1}{10}, \frac{2}{5}, \frac{7}{10}\right).$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE

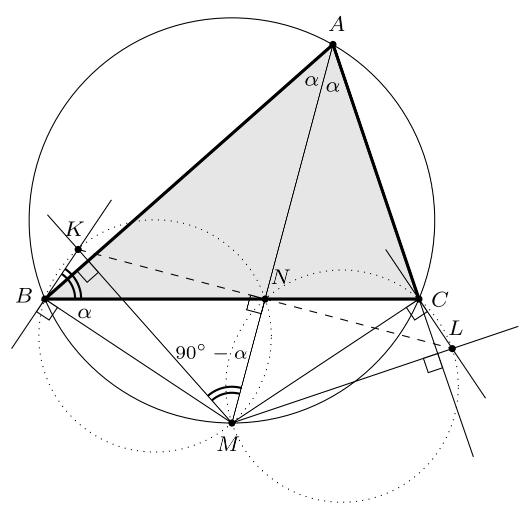
BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICARDO BARROSO CAMPOS, University of Seville, Seville, Spain; MATEI COICULESCU, East Lyme High School, East Lyme, CT, USA; GREG COOK, student, Angelo State University, San Angelo, TX, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; PRITHWIJIT DE, Homi Bhabha Centre for Science Education, Mumbai, India; LEONARD GIUGIUC, Romania; JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; KATHLEEN E. LEWIS, University of the Gambia, Brikama, Gambia; SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; ANGEL PLAZA, University of Las Palmas de Gran Canaria, Spain; SKIDMORE COLLEGE PROBLEM SOLVING GROUP, Skidmore College, Saratoga Springs, NY, USA; DIGBY SMITH, Mount Royal University, Calgary, AB; IRINA STALLION, Southeast Missouri State University, Cape Girardeau, MO, USA; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ITACHI UCHIHA, Hong Kong, China; DANIEL VĂCARU, Piteşti, Romania; STAN WAGON, Macalester College, St. Paul, MN, USA; HAOHAO WANG and JERZY WOJDYLO, Southeast Missouri State University, Cape Girardeau, Missouri, USA; and the proposer.

The proposer points out that if we merely require that $a+b+c+d$ be an integer, then we get exactly two more solutions $(a,b,c,d) = (0,0,0,0), (-9/10, -3/10, 3/10, 9/10)$.

3756. [2012 : 242, 244] *Proposed by Michel Bataille, Rouen, France.*

Let triangle ABC be inscribed in circle Γ and let M be the midpoint of the arc BC of Γ not containing A. The perpendiculars to AB through M and to MB through B intersect at K and the perpendiculars to AC through M and to MC through C intersect at L. Prove that the lines BC, AM intersect at the midpoint of KL.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain, modified by the editor.



Let $N = BC \cap AM$, and $\alpha = \frac{\angle BAC}{2}$. We can assume without loss of generality that $\angle CBA < \angle ACB$ as in the figure (or use directed angles). Because M is the midpoint of the arc BC,

$$\alpha = \angle BAM = \angle MAC = \angle MBC.$$

Using the right angles first at the intersection of KM and AB and then at B, we have

$$\angle NMK = \angle AMK = 90^\circ - \alpha \quad \text{and} \quad \angle NBK = \angle MBK - \angle MBC = 90^\circ - \alpha.$$

From $\angle NMK = \angle NBK$ and $\angle MBK = 90^\circ$ we deduce that $KBMN$ is inscribed in a circle whose diameter is MK, which makes $\angle MNK = 90^\circ$ also. Analogously, $LCNM$ is cyclic with diameter ML and $\angle MNL = 90^\circ$. Because MN is perpendicular to both NK and NL, N must lie on the line KL. Also, the right triangles KNM and LNM have corresponding angles of $90^\circ - \alpha$ at their common vertex M and they share the side MN; consequently, they are congruent, whence $NK = NL$. That is, N is the midpoint of KL.

Also solved by CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; KEE-WAI LAU, Hong Kong, China; MADHAV R. MODAK, formerly of Sir Parashurambhau College, Pune, India; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; ERCOLE SUPPA, Teramo, Italy; EDMUND SWYLAN, Riga, Latvia; ITACHI UCHIHA, Hong Kong, China; JACQUES VERNIN, Marseille, France; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete submission.

3757. [Correction, 2012 : 284, 286] *Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia.*

Let A, B, C be the angles (measured in radians), R the circumradius and r the inradius of a triangle. Prove that

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \le \frac{9}{2\pi} \cdot \frac{R}{r}\,.$$

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Consider the function $f(x) = \ln\left(\frac{\sin\frac{x}{2}}{x}\right) = \ln\left(\sin\frac{x}{2}\right) - \ln x$, $x \in (0,\pi)$. Straightforward computations show that $f'(x) = \frac{1}{2}\cot\frac{x}{2} - \frac{1}{x}$ and

$$f''(x) = -\frac{1}{4}\csc^2 x + \frac{1}{x^2} = \frac{\sin^2\left(\frac{x}{2}\right) - \frac{1}{4}x^2}{x^2\sin^2\left(\frac{x}{2}\right)} = \frac{\left(\sin\frac{x}{2} + \frac{x}{2}\right)\left(\sin\frac{x}{2} - \frac{x}{2}\right)}{x^2\sin^2 x}\,.$$

Since $0 < \sin\frac{x}{2} < \frac{x}{2}$ for $x \in (0,\pi)$ we have $f''(x) < 0$ so f is concave on $(0,\pi)$. Hence, by Jensen's Inequality we have

$$\frac{1}{3}f(A) + \frac{1}{3}f(B) + \frac{1}{3}f(C) \le f\left(\frac{A+B+C}{3}\right) = f\left(\frac{\pi}{3}\right)$$

so

$$\frac{1}{3}\left(\ln\left(\frac{\sin\frac{A}{2}}{A}\right) + \ln\left(\frac{\sin\frac{B}{2}}{B}\right) + \ln\left(\frac{\sin\frac{C}{2}}{C}\right)\right) \le \ln\left(\frac{\sin\frac{\pi}{6}}{\frac{\pi}{3}}\right) = \ln\left(\frac{3}{2\pi}\right)$$

or

$$\ln\left(\frac{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}{ABC}\right)\le\ln\left(\frac{3}{2\pi}\right)^3$$

so

$$\frac{1}{ABC}\le\frac{27}{8\pi^3}\cdot\frac{1}{\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}}\,. \tag{1}$$

It is well known that

$$\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}=\frac{r}{4R}\,. \tag{2}$$

From (1) and (2) we then have

$$\frac{1}{ABC}\le\frac{27}{2\pi^3}\cdot\frac{R}{r}\,. \tag{3}$$

Finally, using (3) and the obvious inequality $AB+BC+CA\le\frac{1}{3}(A+B+C)^2$ we have

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C}=\frac{AB+BC+CA}{ABC}\le\frac{(A+B+C)^2}{3ABC}=\frac{9}{2\pi}\cdot\frac{R}{r}$$

and our proof is complete. Clearly equality holds if and only if $A=B=C$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MARIAN DINCĂ, Bucharest, Romania; NERMIN HODŽIĆ, Dobošnica, Bosnia and Herzegovina and SALEM MALIKIĆ, student, Simon Fraser University, Burnaby, BC; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Arslanagić gave a similar proof and pointed out that (1) is actually inequality 6.59 on p. 188 of the book "Recent Advances in Geometric Inequalities" *(Kluwer Academic Publishers, Dordrecht/Boston/London, 1989) by D.S. Mitrinović, J.E. Pečarić and V. Volenec. We decided to give a proof for completeness. Dinca pointed out that since $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\ge\frac{9}{A+B+C}=\frac{9}{\pi}$ by the AM-HM inequality, the result can be strengthened to a double inequality*

$$\frac{9}{\pi}\le\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\le\frac{9}{2\pi}\cdot\frac{R}{r}$$

or

$$2r\le\frac{2\pi r}{9}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\le R$$

which is a refinement of the famous Euler inequality $2r\le R$.

3758. [2012 : 242, 244] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given a point X on the segment BC, construct a point A such that the incircle of triangle ABC touches BC at X, and that the line joining the Gergonne point and the Nagel points of the triangle is parallel to BC.

Composite of solutions by Daniel Văcaru, Piteşti, Romania; and by the proposer.

We assume the desired triangle ABC exists and let $a=BC$, $b=CA$, $c=AB$, and $s=\frac{1}{2}(a+b+c)$. If X and Z are the points where the incircle touches the sides BC and BA, then (by definition) the cevians AX and CZ intersect in the Gergonne point G_e. From the standard properties of incircles we know that

$BX = BZ = s - b$, $AZ = s - a$, and $CX = s - c$. Because we have been given the segment BC and a point X on it, we therefore know the lengths $s - b = BX$ and $s - c = CX$ and must find $s - a$. Applying Menelaus's theorem to triangle ABX with transversal CG_eZ we have

$$-1 = \frac{AG_e}{G_eX} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = \frac{AG_e}{G_eX} \cdot \frac{s-c}{-a} \cdot \frac{s-b}{s-a},$$

whence,

$$\frac{AG_e}{G_eX} = \frac{a(s-a)}{(s-b)(s-c)}. \qquad (1)$$

We next let X' and Y' be the points where the excircles touch the sides BC and AC, so that the cevians AX' and BY' intersect (by definition) in the Nagel point N_a. Again we know that $BX' = AY' = s-c$, $CX' = s-b$, and $CY' = s-a$. Applying Menelaus's theorem to triangle $AX'C$ with transversal BN_aY', we have

$$-1 = \frac{AN_a}{N_aX'} \cdot \frac{X'B}{BC} \cdot \frac{CY'}{Y'A} = \frac{AN_a}{N_aX'} \cdot \frac{-(s-c)}{a} \cdot \frac{s-a}{s-c},$$

whence,

$$\frac{AN_a}{N_aX'} = \frac{a}{s-a}. \qquad (2)$$

Because the transversal G_eN_a of the triangle AXX' is parallel to XX' if and only if $\frac{AG_e}{G_eX} = \frac{AN_a}{N_aX'}$, we deduce from (1) and (2) that G_eN_a is parallel to BC (which contains the segment XX') if and only if

$$(s-a)^2 = (s-b)(s-c). \qquad (3)$$

Consequently, we want $s - a$ to be the geometric mean of $s - b$ and $s - c$, a quantity whose construction was given by Euclid. From this we get the lengths $b = (s-a)+(s-c)$ and $c = (s-a)+(s-b)$. Because our argument is reversible, as long as the quantities a, b, c satisfy the triangle inequality, there will be a unique triangle ABC that satisfies (3), whose incircle touches BC at the given point X. Here, then, is its construction.

1. Construct the perpendicular to BC at X, and call P either point where it intersects the circle whose diameter is BC. (Then PX is the geometric mean of $BX = s - b$ and $CX = s - c$; that is, $PX = \sqrt{(s-b)(s-c)} = s - a$.)

2. Call B' and C' the points where the circle with centre X and radius XP intersects the line BC, labeled so that B and C' are on the same side of X. (Then $BB' = BX + XB' = s - b + s - a = c$ and $C'C = C'X + XC = s - a + s - c = b$.)

3. The desired third vertex A will be either point where the circle with centre B and radius $c = BB'$ intersects the circle with centre C and radius $b = CC'$.

It remains to verify that the circles in the third step of the construction will always intersect; that is, we must show that the constructed segments a, b, c satisfy the triangle inequality:

$$\begin{aligned} c+b &= BB'+CC' = (BX+XB')+(CX+XC') > BX+XC = a, \\ a+b &= (BX+XC)+(CX+XC') > BX+XC' = BX+XB' = c, \\ a+c &= (BX+XC)+(BX+XB') > CX+XB' = CX+XC' = b. \end{aligned}$$

This proves the existence of the constructed triangle ABC whose incircle touches BC at the given point X and whose sides satisfy equation (3), so that $G_eN_a||BC$ as required.

Also solved by OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; and TITU ZVONARU, Comăneşti, Romania.

3759. [2012 : 242, 244] *Proposed by Nguyen Minh Ha, Hanoi, Vietnam.*

Given a convex polygon $A_1A_2\cdots A_n$ with an interior point P. Let $a_i = \sum_{j=1}^{n} A_iA_j$. Prove that $\sum_{i=1}^{n} PA_i < \max_{1\le j\le n}\{a_j\}$.

Solution by Oliver Geupel, Brühl, NRW, Germany.

The position vector $\vec{P}$ of a point P can be expressed as a convex linear combination of the position vectors of the A_i's, $i = 1, 2, \ldots, n$

$$\vec{P} = \sum_{i=1}^{n} \lambda_i \vec{A}_i$$

where $\lambda_i > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. Hence,

$$\vec{A}_i - \vec{P} = \vec{A}_i - \sum_{j=1}^{n} \lambda_j \vec{A}_j = \sum_{j=1}^{n} \lambda_j(\vec{A}_i - \vec{A}_j), \quad i = 1, 2, \ldots, n.$$

By the triangle inequality, we then have, for each $i = 1, 2, \ldots, n$,

$$|PA_i| = \left|\sum_{j=1}^{n} \lambda_j(\vec{A}_i - \vec{A}_j)\right| < \sum_{j=1}^{n} \lambda_j |A_jA_i|. \tag{1}$$

[*Ed. : For clarity we use $|XY|$ to denote the distance between points X and Y, that is, the length of the vector $\vec{Y} - \vec{X}$.*]

Note that the inequality in (1) is strict since the vectors $\vec{A}_i - \vec{A}_j$ are not collinear.

Adding the inequality in (1) over $i = 1, 2, \ldots, n$ we then obtain

$$\sum_{i=1}^{n} |PA_i| < \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j |A_j A_i| = \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} |A_j A_i|$$
$$= \sum_{j=1}^{n} \lambda_j a_j \le \max_{1 \le j \le n} \{a_j\}.$$

This completes the proof.

Also solved by the proposer.

Geupel remarked that the problem is a generalization of problem **2215** *[1997 : 109; 1998 : 121] which dealt with the case $n = 3$. This same special problem was also posed in the internet forum* Art of Problem Solving. *The solution by a solver nicknamed* gemath *straightforwardly generalizes to his proof featured above. He gave the following reference:*

`http://www.artofproblemsolving.com/forum/viewtopic.php?p=634149.`

3760. [2012 : 243, 244] *Proposed by Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $p \ge 2$ be an integer. Determine the limit

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{\sqrt[p]{n}}{\sum_{j=1}^{p} \sqrt[p]{k^j (n+k)^{p-j+1}}}.$$

Solution by Haohao Wang and Jerzy Wojdylo, Southeast Missouri State University, Cape Girardeau, Missouri, USA.

We claim that

$$\lim_{n\to\infty} \sum_{k\ge 1} \frac{\sqrt[p]{n}}{\sum_{j=1}^{p} \sqrt[p]{k^j \left(n+k\right)^{p-j+1}}} = \frac{p}{p-1}.$$

To prove our claim, we note that since $a + a^2 + \cdots + a^p = \dfrac{a\left(1-a^p\right)}{1-a}$, we have

$$\begin{aligned}
\sum_{j=1}^{p} \sqrt[p]{k^j \left(n+k\right)^{p-j+1}} &= \left(n+k\right)^{(p+1)/p} \left[\sum_{j=1}^{p} \left(\frac{k}{n+k}\right)^{j/p}\right] \\
&= \left(n+k\right)^{(p+1)/p} \cdot \frac{\left(\frac{k}{n+k}\right)^{1/p} \left(1 - \frac{k}{n+k}\right)}{1 - \left(\frac{k}{n+k}\right)^{1/p}} \\
&= \frac{\left(n+k\right)^{(p+1)/p} \left(\frac{k}{n+k}\right)^{1/p} \frac{n}{n+k}}{1 - \left(\frac{k}{n+k}\right)^{1/p}} \\
&= \frac{nk^{1/p}}{1 - \left(\frac{k}{n+k}\right)^{1/p}}.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\sqrt[p]{n}}{\sum_{j=1}^{p}\sqrt[p]{k^j\,(n+k)^{p-j+1}}} = \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{\sqrt[p]{n}\left[1-\left(\frac{k}{n+k}\right)^{1/p}\right]}{nk^{1/p}}$$

$$= \lim_{n\to\infty} n^{(1-p)/p}\cdot\sum_{k=1}^{\infty}\left[k^{-1/p}-(n+k)^{-1/p}\right]$$

$$= \lim_{n\to\infty} n^{(1-p)/p}\cdot\sum_{k=1}^{n}k^{-1/p}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{-1/p}$$

$$= \int_0^1 x^{-1/p}\,dx$$

$$= \frac{p}{p-1},$$

as claimed.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; ANASTASIOS KOTRONONIS, Athens, Greece; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer. Three incorrect solutions were received.

Note that this problem is a generalization of problem Q1011 ***Math. Mag.*** *84(3), 2011, p. 230.*